PROBLEM SET 10: SOLUTIONS

5.5.2 (skip the question about geodesics)

To see that the map in the textbook gives the unrolling sketched there, note that for any fixed t, the image lies on the cone $(u' \cos v', u' \sin v', bu')$. Then observe that u' will range from 0 to $+\infty$, while v' from 0 to $2\pi/\sqrt{1+ta^2}$, which means it will be exactly a wedge aprt of that cone.

To see that for a fixed t, the map I is an isometry, compute E, F, and G for the equation $\vec{y}(u,v)$ defining a parameterization of the new cone by u and v and check that these are the same E, F, and G as for the original cone. Here is this complutation for E.

$$\vec{y}_u = (a' \cos(v/a'), a' \sin(v/a'), a\sqrt{1-t}),$$

where $a' = \sqrt{1+ta^2}$. Then

$$E = \vec{y}_u \cdot \vec{y}_u = (a')^2 + a^2(1-t) = 1 + a^2.$$

A similar simpler computation shows that for the original cone $E = 1 + a^2$.

5.5.4 (skip the question about geodesics) The unrolling map will be taking a point $(u, \cos v, \sin v)$ on the cylinder to a point on the segment joining the point on the cone with the point $(u, v, 0)$ of the xy plane. The corresponding equation is

$$(u, (1-t)\cos v + tv, (1-t)\sin v).$$

You are not asked to see it is an isometry for all values of t. In fact, it is an isometry only at $t = 0$ and 1.

5.5.9 (do it only for the helicoid) Compute $\vec{U} = \vec{x}_u \times \vec{x}_v / \sqrt{EG - F^2}$, where $E = \vec{x}_u \cdot \vec{x}_u = 1$, $F = 0$, and $G = 1 + u^2$ are also obtained from computations (as in Section 3.2). Then for $\vec{U}(u,v) = (\sin v, cosv, u)/\sqrt{1+u^2}$, its own $E' = (1+u^2)^{-2}$, $F' = 0$, and $G' = (1+u^2)^{-1}$, which differ from E, F, and G by a common factor of $(1+u^2)^{-2}$. Thus the scaling factor is $1 + u^2$.

5.5.10 The first part of the problem is done just as the previous problem. We discussed the second one in class.

6.2.1 $\nabla^{\mathbb{R}^3}_{\alpha'} \alpha' = \frac{d\alpha'}{ds}$ by definition. This gives α''.

6.2.3 (only 1 and 2) I will do only (2): (1) is similar. First, prove it for $\nabla^{\mathbb{R}^3}$: $\nabla^{\mathbb{R}^3}_{\alpha'}(fZ) = \frac{d(fZ)(\alpha)}{dt} = \frac{d(f(\alpha)Z(\alpha))}{dt} = \frac{d(f)(\alpha)}{dt} Z(\alpha) + f \frac{d(Z)(\alpha)}{dt} = \frac{d(f \circ \alpha)}{dt} Z + f \nabla^{\mathbb{R}^3}_{\alpha'} Z$. To get $\nabla_{\alpha'}$, apply the projection operator P to the tangent plane along the normal direction: $\nabla_{\alpha'}(fZ) = P(\nabla^{\mathbb{R}^3}_{\alpha'}(fZ)) = P(\frac{d(f \circ \alpha)}{dt} Z + f \nabla^{\mathbb{R}^3}_{\alpha'} Z) = \frac{d(f \circ \alpha)}{dt} P(Z) + f P(\nabla^{\mathbb{R}^3}_{\alpha'} Z)$, because P is linear, and f is a scalar. Then continue as

$$\frac{d(f \circ \alpha)}{dt} Z + f \nabla_{\alpha'} Z,$$

because Z was assumed to be a tangent vector to the surface.

1. Prove that if a curve $\alpha(t)$ on a surface M is both a line of curvature (i.e., $\alpha'(t)$ is an eigenvector of the shape operator, see p. 81) and a geodesic, then $\alpha(t)$ is a plane curve.

Solution. Reparameterize α by arclength. If it is a geodesic, then $\alpha'' = \kappa N$ is parallel to the surface normal $\vec{U}$. This means that the unit normal N to the curve is $\pm\vec{U}$, because both have a unit length.

If α is a line of curvature, then α' is an eigenvector of the shape operator: $-\nabla^{\mathbb{R}^3}_{\alpha'} U = k\alpha' = kT$, but $\nabla^{\mathbb{R}^3}_{\alpha'} U = dU/ds = \pm dN/ds = \pm(-\kappa T + \tau B)$. This implies $-k = \pm\kappa$ and $0 = \tau$, which means the curve is planar.

2. Show that if a geodesic is a plane curve, then it is a line of curvature.

Solution. If it is a plane curve, its torsion is zero, and we can reverse the argument above.

3. Give an example of a line of curvature which is a plane curve and not a geodesic.

Solution. In a plane the unit normal is constant and therefore shape operator is zero. Thus, every tangent vector is an eigenvector of the shape operator. Thus, every curve is a line of curvature. However, only straight lines are geodesics.