PROBLEM SET 11: SOLUTIONS

6.3.2 $d/ds(V \cdot \alpha') = \nabla_{\alpha'} V \cdot \alpha' + V \cdot \nabla_{\alpha'} \alpha' = 0 \cdot alpha' + V \cdot 0 = 0$. Similarly, $d/ds(V \cdot V) = 0$ and $d/ds(\alpha' \cdot \alpha') = 0$. This means $V \cdot \alpha'$, $|V|$, and $|\alpha'|$ are constant, and therefore, $\cos\phi = V \cdot \alpha'/(|V||\alpha'|)$ is also constant. Thus so is ϕ.

For the angle between V and W, it is the same story: you show that $V \cdot W$ is constant.

If the holonomy of V along a curve is $\theta(1) - \theta(0)$ and the angle between V and W is ϕ, which is constant along the curve by the above, then the holonomy of W along the curve will be $\theta(1) + \phi - (\theta(0) + \phi) = \theta(1) - \theta(0)$.

5.1.2 The topmost parallel is a circle $\alpha(s)$ of radius a, therefore, $\kappa = 1/a$. α' will be the tangent to this circle, while $\alpha'' = \kappa n = n/a$, where n is the unit normal to the circle, will point to the center. $\alpha'' \times \alpha'$ will then point in the direction of negative z axis, by the thumb rule. A unit exterior normal to the torus at the topmost parallel points in the direction of positive z axis. Thus $\theta = \pi$ and $\cos\theta = -1$ and $\kappa_g = \kappa\cos\theta = -1/a$. The normal curvature is computed using the angle between the normal to the curve and the normal to the surface at this curve. This angle is $\pi/2$, whose cosine is zero. Thus $k = \kappa \cdot 0 = 0$, and the equation $\kappa^2 = k^2 + \kappa_g^2$ is equivalent to $1/a^2 = 0 + 1/a^2$, which is true.

5.1.3 Let T be the unit tangent to α, a curve on an (oriented, as usually) surface, U the unit normal to the surface, which is of course orthogonal to T. Then T, U, and $T \times U$ form an orthonormal frame, and what we need to do is to decompose α'' in this frame: $\alpha'' = aT + bU + cT \times U$. Taking the dot products of this equation with the elements of the orthonormal frame, we find $a = \alpha'' \cdot T$, $b = \alpha'' \cdot U$, and $c = \alpha'' \cdot T \times U$. This already gives the required formula for b. It remains to find the dot products for a and c.

Let s be the arclength parameter $s = \int_{t_0}^{t} |\alpha'(t)| dt$ and $\nu = ds/dt = |alpha'|$. $T = d/ds\alpha(t(s)) = \alpha' \cdot dt/ds = \alpha'/\nu$. Differentiate $\alpha' \cdot \alpha' = \nu^2$. We will get $2\alpha'' \times \alpha' = 2\nu d\nu/dt$. Plug in $\alpha' = \nu T$ into the left-hand side to get $a = \alpha'' \cdot T = d\nu/dt$.

Now, $\alpha'' \cdot T \times U = \alpha'' \cdot \frac{\alpha'}{\nu} \times U = \alpha'' \times \alpha' \cdot U/\nu$, because both are the determinants of the same 3×3 matrix up to an even row permutation. Now it is time to compute α'': we have $d/ds(\alpha') = \alpha'' dt/ds = \alpha''/\nu$. Recall our previous computations: $\alpha' = \nu T$, whence on the other hand $d/ds(\alpha') = (d\nu/ds)T + \nu dT/ds = (d\nu/ds)T + \nu\kappa N$, where N is the normal to the curve. Combining this with the previous computation of $d/ds\alpha'$, we get $\alpha'' = \nu((d\nu/ds)T + \nu\kappa N)$. Thus $c = \alpha'' \times \alpha' = \nu^2((d\nu/ds)T \times T + \nu\kappa N \times T) = \nu^3\kappa N \times T$. Now $\alpha'' \times \alpha' \cdot U/\nu = \nu^2\kappa|N \times T||U|\cos\theta$, where θ is the angle between $N \times T$ and U. Continuing the computation, we get $c = \nu^2\kappa_g$, because N, T, and U are unit vectors, and $\kappa_g = \kappa\cos\theta$.

5.1.6 (the "if" part) If α lines in a plane P perpendicular to M anywhere their intersection, then the curve normal is in P. Since the curve normal N is perpendicular to the curve tangent, N will be parallel to the surface normal U, i.e. $N = \pm U$. Therefore α is a geodesic.

$S_p\alpha' = -\nabla_{\alpha'}U = \pm\nabla_{\alpha'}N = \pm N' = \pm\kappa T \pm \tau N = \pm\kappa T = \pm\kappa\alpha'$, because $\tau = 0$ as α is a plane curve.

5.2.1 For the cylinder $x = (\cos u, \sin u, v)$ and $x_u = (-\sin u, \cos u, 0)$, $x_v = (0, 0, 1)$, and $E = 1$, $F = 0$, $G = 1$. The geodesic equations become $u'' = v'' = 0$, which solve as $u = at + b$ and $v = ct + d$, which describe all possible lines in the uv plane. Identifying these lines on the cylinder, we see that the geodesics are the vertical lines, the horizontal circles, and the helices.

5.2.2 Differentiating $\alpha' \cdot \alpha' = c$, we get $2u''u'E + (u')^2E_uu' + (u')^2E_vv' + 2v''v'G + (v')^2G_uu' + (v')^2G_vv' = 0$. Replacing u'' by the first geodesic equation, we get the second one after all cancellations.

Reversing these calculations means using the two geodesic equations to show that the above equation, which is equivalent to $(\alpha' \cdot \alpha')' = 0$, is satisfied. Thus, the speed of a geodesic is constant.

5.2.3 The geodesics are lines of curvature, therefore, their tangent vectors are eigenvectors of the shape operator at all points. Since in each tangent direction at each point of the surface, there goes a geodesic, we see that all tangent vectors are eigenvectors of the shape operator. Then from linear algebra, at each point of the surface, the eigenvalues must be all equal (If you do not know this fact, it is a simple exercise: show that if all vectors are eigenvectors for a matrix, then the corresponding eigenvalues are all equal), which means the principal curvatures are equal. Therefore, each point is umbilc. Then by Theorem 3.5.1 M is part of a plane or a sphere.

1. For a torus, the parallels $u = 0$, $u = \pi/2$, and $u = \pi$ are called the maximum parallel, the topmost parallel, and the minimum parallel, respectively. Check which of these parallels are geodesics and which are lines of curvature.

Solution. The maximum and the minimum parallels are geodesics, because their normals are obviously parallel to the normal to the torus. The topmost parallel is not a geodesic , because its normal is horizontal, while the normal to the torus along it is vertical.

The maximum and minimum parallels are lines of curvature, because they are plane geodesics, see a problem from the previous homework. The topmost parallel α is a line of curvature, because the surface normal $\vec{N}$ along it is vertical and thereby constant. Therefore, its covariant derivative in any direction is zero. In particular, $S_p(\alpha') = -\nabla_{\alpha'}\vec{N} = 0 = 0 \cdot \alpha'$. By definition, this means $alpha$ is a line of curvature.

2. Intersect the cylinder $x^2 + y^2 = 1$ with a plane passing through the x axis and making an angle θ with the xy plane. Show that the intersecting curve is an ellipse C. Compute the geodesic curvature of C in the cylinder at the points where C meets its axes.

Solution. These will four points on this picture: a top one, two middle ones and a bottom one. The geodesic curvature at the middle ones will be zero, because the normal to the ellipse at these points will point to its center, which is on the central axis of the cylinder and therefore parallel to the normal of the cylinder. The geodesic curvature of the top and the bottom ones is obtained by computing the curvature of the ellipse at such a point and multiplying it by the sine of the angle between the curve normal and the surface normal, which is the same as θ. Thus, the answer will be $\kappa \sin\theta$, where κ is the curvature of the ellipse at a point where it meets its long axis, and we have to find it.

Parameterize the ellipse by $u = x$ and $v = y/\cos\theta$ — this is a parameterization of the slanted plane in which the uv distance will be the same as the distance in space. In this plane, the ellipse will be given by the equation $u^2 + v^2\cos^2(\theta) = 1$, in which we recognize an ellipse, by the way. Let $b = 1/\cos(\theta)$ and parameterize the ellipse $\alpha(t)$ by $u = \cos t$, $v = b\sin t$. Then we need to find the curvature of this curve at $t = \pi/2$. $u' = -\sin t$, $v' = b\cos t$, and $(\alpha')^2 = \sin^2 t + b^2\cos^2 t$, which means it is not the arclength. To find the curvature, we have to pass to an arclength parameter $s = \int|\alpha'(t)|dt$. Note that $ds/dt = |\alpha'|$, which will be useful later. Notice also that at $t = \pi/2$, $(\alpha')^2 = 1$ and $ds/dt = 1$. Let us find T and dT/ds, and we will get $\kappa = |dT/ds|$. Then $\alpha' = d\alpha/dt = (d\alpha/ds)(ds/dt) = T|\alpha'|$. We have computed α' and its length above. This will allow us to find T as a function of t. Take the derivative of T in t and compute it through the derivative in s: $dT/dt = (dT/ds)(ds/dt)$ This will produce dT/ds and finally its length, which is κ. The answer is $\kappa = \cos^2\theta$ at $t = \pi/2$, and $\kappa_g = \cos^2\theta\sin\theta$.