SOLUTION OF PROBLEM 10 FROM THE SECOND SAMPLE SET

Since the curve is a geodesic, its unit normal N is equal to the unit normal U to the surface. (In general, this is true up to a sign, but we can change the orientation on the surface, to make $N = U$.) Using one of the Frenet formulas, we get $U' = N' = -\kappa T + \tau B$. Thus $U' \cdot U' = \kappa^2 + \tau^2$. Let us compute $U' \cdot U'$, using the first and the second fundamental forms. If we show $U' \cdot U' = -K$, we are done.

Use the chain rule: $U' = U_u u' + U_v v'$. We computed the partials U_u and U_v (which were interesting to us, because they are equal to $-S_p(x_u)$ and $-S_p(x_v)$, respectively), when we obtained the Weingarten equations and the formulas for the mean and the Gaussian curvatures through the first and the second fundamental forms. There we introduced a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

so that

$$\begin{aligned} U_u &= a_{11} x_u + a_{21} x_v, \\ U_v &= a_{12} x_u + a_{22} x_v. \end{aligned}$$

In other words, this matrix was defined by these equations. In the matrix form, this equation may be rewritten as follows.

$$\begin{pmatrix} U_u \\ U_v \end{pmatrix} = A^T \begin{pmatrix} x_u \\ x_v \end{pmatrix}$$

Our chain rule above in the matrix form looks like

$$U' = (U_u \quad U_v) \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Then

$$\begin{aligned} U' \cdot U' = (U')^T U' &= (u' \quad v') \begin{pmatrix} U_u \\ U_v \end{pmatrix} (U_u \quad U_v) \begin{pmatrix} u' \\ v' \end{pmatrix} \\ &= (u' \quad v') A^T \begin{pmatrix} x_u \\ x_v \end{pmatrix} (x_u \quad x_v) A \begin{pmatrix} u' \\ v' \end{pmatrix} \\ &= (u' \quad v') A^T \,\mathrm{I}\, A \begin{pmatrix} u' \\ v' \end{pmatrix}, \end{aligned}$$

where I is the first fundamental form

$$\mathrm{I} = \begin{pmatrix} x_u \\ x_v \end{pmatrix} (x_u \quad x_v) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

We computed this matrix A by obtaining the equation

$$-\,\mathrm{II} = A^T \,\mathrm{I},$$

where

$$\mathrm{II} = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

is the second fundamental form. Thus we can replace $A^T I$ by $-\,\mathrm{II}$ in the equation for $U' \cdot U'$ above:

$$U' \cdot U' = -(u' \quad v')\,\mathrm{II}\,A \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Since the fundamental forms are symmetric, $\mathrm{II} = \mathrm{II}^T = -\,\mathrm{I}^T A = -\,\mathrm{I}\,A$, and we continue as follows.

$$U' \cdot U' = (u' \quad v')\,\mathrm{I}\,A^2 \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Now recall that $A^2 - 2HA + K\,\mathrm{Id} = 0$ (the characteristic equation of the matrix A). The surface is minimal, so $H = 0$ and we have $A^2 = -K\,\mathrm{Id}$. Finally,

$$U' \cdot U' = -K(u' \quad v')\,\mathrm{I} \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

But $x' = x_u u' + x_v v'$ or in the matrix form

$$x' = (x_u \quad x_v) \begin{pmatrix} u' \\ v' \end{pmatrix}$$

and

$$x' \cdot x' = (x')^T x' = (u' \quad v')\,\mathrm{I} \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

$x' \cdot x' = 1$, because $x(s)$ is the equation of the geodesic, which we parameterized by arclength, as usually. Thus,

$$U' \cdot U' = -K.$$