SAMPLE FINAL EXAM - MATH 5378, SPRING 2013

THIS WILL BE A CLOSED-BOOK, CLOSED-NOTES EXAM. YOU CAN USE IN YOUR SOLUTIONS ANY RESULT THAT WAS COVERED IN CLASS OR BY THE TEXT. YOU CAN ALSO USE ANY OF THE RESULTS FROM THE HOMEWORK.

Complicated formulas, such as the geodesic equations

$$u'' + \Gamma_{11}^1 (u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 (v')^2 = 0,$$
$$v'' + \Gamma_{11}^2 (u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 (v')^2 = 0$$

and the Gauss and Mainardi-Codazzi equations (see p. 178 of the text), will be provided on the actual exam.

(1) Assume $\alpha(s)$ is a unit speed curve of nonzero constant curvature whose normal lines are all perpendicular to a given constant vector. Show that α is part of either a circle or a standard circular helix, which is a curve $\beta(t)$ such that, after a Euclidean motion of the space, may be represented by an equation $\beta(t) = (a\cos t, a\sin t, bt)$, for $a \neq 0, b \neq 0$.

(2) Show that a space curve with $\kappa(s) \neq 0$, $\tau(s) \neq 0$ and $\kappa'(s) \neq 0$ for all s lies on a sphere if and only if

$$\left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2 = \text{const.}$$

(3) Compute the first fundamental form of the following surfaces:
  (a) $x(u,v) = \left(\frac{a}{2}\left(u + \frac{1}{u}\right), \frac{b}{2}\left(u - \frac{1}{u}\right), v\right)$ (the hyperbolic cylinder);
  (b) $x(s,t) = \alpha(s) + tB(s)$ (the binormal surface of a curve unit-speed curve $\alpha(s)$ with a binormal $B(s)$).

(4) Show that a compact, regular, oriented surface M in $\mathbb{R}^3$ has a point with $K > 0$. Furthermore, show that if M is not homeomorphic to a sphere, then it also has a point with $K < 0$ and a point with $K = 0$.

(5) Show there is no surface in $\mathbb{R}^3$ with $E = G = e = 1$, $F = f = 0$ and $g = -1$.

(6) Let $\beta : I \longrightarrow \mathbb{R}^3$ be a curve of nonzero curvature and let $x(u,v) = \beta(u) + v\beta'(u)$ be its tangent surface. Show that it is locally isometric to a cone (without the vertex).

(7) Compute the second fundamental form, the Gaussian and the mean curvatures of the following surface: $xyz = a^3$, $a \neq 0$.

(8) Find the angle through which the tangent vector to a closed curve α on the unit sphere $x^2 + y^2 + z^2 = 1$ turns, if
  (a) α is a parallel;
  (b) α consists of two meridians different by an angle ϕ and the part of the equator between them.

(9) Prove that on a surface of revolution, a meridian is always a geodesic directly, without using Clairaut's relation. Assume the equation of a general

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surface of revolution given, as well as the formulas for the Christoffel symbols of a surface of revolution and the geodesic equations.

(10) Let M be a surface in $\mathbb{R}^3$ such that all its geodesics are plane curves. Show that M is part of either a plane, or a sphere.

(11) Find geodesics on a general cylindrical surface

$$x(u, v) = (f(u), g(u), v),$$

where u is the arclength parameter of the base curve $(f(u), g(u))$.

(12) Show that the equations $x_1^2 + x_2^2 = 1$ and $x_3^2 + x_4^2 = 1$ in $\mathbb{R}^4$ define a smooth manifold of dimension 2.