MATH 5616H: HONORS ANALYSIS
SAMPLE FINAL EXAM

INSTRUCTOR: SASHA VORONOV

You may not use a calculator, notes, books, etc. Only the exam paper, scratch paper, and a pencil or pen may be kept on your desk during the test. You must show all work.

Good luck!

Problem 1. For $x \in [0,1]$ define f_n by the formula

$$f_n(x) = \frac{3nx}{1+n^2x^2}.$$

Prove that the family $\{f_n\}$ is not equicontinuous. [*Equicontinuous* is the same thing as *uniformly equicontinuous.*]

Problem 2. Prove that if U and V are open subsets of $\mathbb{R}$, then $U \times V$ is an open subset of $\mathbb{R}^2$.

Problem 3. Let $A_1, \dots, A_n, \dots$ be nonempty connected compact sets in a metric space M. Suppose $A_{n+1} \subset A_n$ for every n. Show that $\bigcap_{n=1}^{\infty} A_n$ is connected.

Problem 4. Let B, C, and D respectively stand for *bounded*, *continuous*, and *differentiable*. For the three functions f, g, and $h : \mathbb{R}^2 \to \mathbb{R}$ that are 0 at $(0,0)$ and given by $\frac{xy}{x^2+y^2}$, $\frac{xy^2}{x^2+y^2}$, $\frac{x^2y^2}{x^2+y^2}$ elsewhere, say which functions have which properties and why.

Problem 5. For which values of T can we find a unique solution of the ODE $x''(t) = -x(t)$ satisfying the boundary conditions $x(0) = a_1$ and $x(T) = a_2$ for any values of a_1 and a_2?

Problem 6. Prove that if $\mathcal{F}$ is a finite collection of open intervals that covers a compact interval $[a,b]$, then the sum of the lengths of the intervals in the collection is strictly greater than $b-a$.

Problem 7. Suppose $f \in L^1([0,1])$. Prove that $\lim_{\varepsilon \to 0+} \int_{[0,\varepsilon]} f d\mu = 0$.

Problem 8. Let $f * g(x) = \int_{-\pi}^{\pi} f(x-y)g(y)dy$ be the convolution of two continuous periodic functions f and g. Give a formula for the Fourier coefficients of $f * g$ in terms of the Fourier coefficients of f and g.

Problem 9. Prove that the collection of all real trigonometric polynomials $a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$, where N is any nonnegative integer and the a_n and b_n are real, is dense in $C([0,\pi])$ with respect to the sup-norm metric.

Date: May 4, 2005.