MATH 5615H: INTRODUCTION TO ANALYSIS I
SAMPLE MIDTERM EXAM I

You may not use notes, books, etc. Only the exam paper, a pencil or pen may be kept on your desk during the test. Calculators are not allowed, either, but will not be needed. Ask me, and I will compute anything for you, if you need me to. Unless stated otherwise, please show all of your work and justify your answers in order to receive full credit.

Good luck!

(1) Suppose you know that the arithmetic roots of nonnegative numbers exist, that is, for each real $a \geq 0$ and natural n, there is a nonnegative real number, denoted $\sqrt[n]{a}$, such that $(\sqrt[n]{a})^n = a$. Using the axioms of an ordered field, prove that $0 \leq a < b$ implies $0 \leq \sqrt[n]{a} < \sqrt[n]{b}$ (a) for $n = 2$, (b) for an arbitrary natural number n.

(2) Show that the set 2^A of all subsets of an infinite set A has greater cardinality than A.

(3) Let A be a closed subset in a metric space X. Prove that $\partial A = A$ if and only if $A^\circ = \emptyset$. Here $\partial A = \{x \in X \mid$ each ball around x contains a point in A and a point not in A$\}$ is the boundary and A° is the interior. Remember that "if and only if" means two statements.

Solution: Assume $\partial A = A$. If $x \in A^\circ$, then there is a ball $B_r(x) \subset A$. This implies that $x \notin \partial A$, but $x \in B_r(x) \subset A$, which is a contradiction. Thus, there are no points in A°.

Now assume $A^\circ = \emptyset$. Show first that $A \subset \partial A$. Indeed, for any point $x \in A$, we know that $x \notin A^\circ$, and thus every ball $B_r(x)$ contains a point not in A. However, the center x of the ball is in A, therefore $x \in \partial A$.

Let us show that $\partial A \subset A$. Indeed, for any point $x \in \partial A$, if $x \notin A$, then x must be a limit point of A by definition of the boundary. Since it is given that A is closed, this limit point x will actually be in A. We get a contradiction: we assumed $x \notin A$ and showed that $x \in A$. Thus, x must be in A. $\square$

(4) Find an example of a connected set in $\mathbb{R}^2$ whose interior is disconnected.

Why not check up all your work?

Date: October 3, 2014; Solution to #3 posted on October 8.