MATH 5615H: INTRODUCTION TO ANALYSIS I
SAMPLE MIDTERM EXAM II

You may not use notes, books, etc. Only the exam paper, a pencil or pen may be kept on your desk during the test. Calculators are not allowed, either, but will not be needed. Ask me, and I will compute anything for you, if you need me to. Unless stated otherwise, please show all of your work and justify your answers in order to receive full credit.

Good luck!

Problem 1. Assume that the sequence $\{a_n\} \subset \mathbb{R}$ converges to a number $a \in \mathbb{R}$. Define the sequence $\{b_n\}$ by

$$b_n := \frac{1}{n} \sum_{k=1}^{n} a_k.$$

Prove that the sequence $\{b_n\}$ converges to the same limit a.

Problem 2. Suppose that $a_n > 0$, and $\sum_{n=1}^{\infty} a_n$ diverges. Prove that the following series must also diverge:

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}.$$

Problem 3. Assume that the function $f : (0, +\infty) \to \mathbb{R}$ is continuous. Assume also that $\lim_{x\to 0+} f(x)$ and $\lim_{x\to+\infty} f(x)$ exist and are finite. Prove that f is bounded on $(0, +\infty)$.

Problem 4. Suppose for some $n \geq 1$, the $n+1$st derivative of f exists and is continuous on some open interval $(-a, a)$ with $a > 0$. Suppose also that there is a polynomial $p(x)$ of degree $\leq n$ such that

$$|f(x) - p(x)| \leq C|x|^{n+1}$$

for some $C > 0$ and all $x \in (-a, a)$. Prove that the polynomial $p(x)$ is the nth Taylor polynomial for f centered at 0. That is, prove that

$$p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

Date: November 30; solution to Problem 4 added on December 3, 2014.