MATH 5615H: ANALYSIS
A BRIEF SOLUTION TO PROBLEM 2 ON HW 4

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Here is a solution of Problem 2 on Homework 4.

Step 1. Construct a collection $\{U_n\}_{n\in\mathbb{N}}$ of open sets in X, possibly with repetitions $U_i = U_j$ for some $i \neq j \in \mathbb{N}$, so that for any open set $V \subset X$ and each $x \in A \cap V$, there is $n \in \mathbb{N}$ such that $x \in U_n \subset V$. (Such a collection is called a *countable base for* A.)

To construct a countable base, first show that for each $n \in \mathbb{N}$, there is a finite number of balls of radius $1/n$ covering A. Indeed, take $x_1 \in A$ and the ball $B_{1/n}(x_1)$ of radius $1/n$ around x_1. If the ball covers A, we are done, stop. If not, there is $x_2 \in A$ outside of the ball, which means $d(x_2, x_1) \geq 1/n$. Take the ball $B_{1/n}(x_2)$. If the union of these two balls covers A, we are done, stop. Otherwise, take $x_3 \in A$ not in the union of these two balls. Note that x_3 is at least at distance $1/n$ from x_1 and x_2. Take the ball $B_{1/n}(x_3)$ and keep going. I claim that this process will not go unboundedly. Indeed, if it did, we would have an infinite set $\{x_1, x_2, \dots\}$ in A. It was given in the problem that each infinite subset of A has a limit point. Let x be a limit point of the above infinite set. Then if we take a ball of sufficiently small radius around x, it will contain at least two different points x_i and x_j (take a ball around x of one radius, find x_i in that ball, then take a ball around x of sufficiently smaller radius and find x_j in it). If we take the radius of the first ball to be less than $1/2n$, then $d(x_i, x_j) < 1/n$, which contradicts the construction of the sequence of points.

Then take as $\{U_i\}_{i\in\mathbb{N}}$ the balls from these finite collections of balls for all $n \in \mathbb{N}$. It is a countable (of finite) collection, because we have taken the union of countably many finite sets, possibly some of them repeating. It is also a countable base. Indeed, you get a countable base, because for each open V and $x \in A \cap V$, there is a ball B of radius $r > 0$ centered at x and contained in V, because V is open. Take n such that $1/n < r/2$. Then, since A is covered by the balls of radius $1/n$ from our collection $\{U_i\}$, the point x will be in one of these balls U_i. The inequality $1/n < r/2$ ensures that the ball U_i of radius $1/n$ is

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contained in the ball B of radius r, and that ball was part of V. Thus, we get $x \in U_i \subset V$.

Step 2. Let us use a countable base $\{U_i\}_{i \in \mathbb{N}}$ that we constructed in Step 1. Given an open covering $\{V_\alpha\}_{\alpha \in J}$ of A, we need to find a countable of finite subcovering. Define $I := \{n \in \mathbb{N} \mid U_n \subset V_\alpha \text{ for some } \alpha \in J\}$. Obviously, $I \subset \mathbb{N}$. Thus, it is countable or finite. Choose one such α for each $n \in I$ and call it $\alpha(n)$. Claim: $\{V_\alpha\}_{\alpha \in I}$ is a countable of finite subcovering of A. We just saw it was countable or finite, it is a subcollection of $\{V_\alpha\}_{\alpha \in J}$ by construction. It remains to see that the subcollection covers A.

For each $x \in A$ there is $\alpha \in J$ such that $x \in V_\alpha$. Apply the countable base property (Step 1) to V_α and $x \in A \cap V_\alpha$, that is to say, find $n \in \mathbb{N}$ such that $x \in U_n \subset V_\alpha$. Then this n is in I by definition of I. Take $\alpha(n)$ corresponding to that n. Then $U_n \subset V_{\alpha(n)}$ by definition of $\alpha(n)$. We also knew that $x \in U_n$. Thus $x \in V_{\alpha(n)}$.