MATH 5615H: ANALYSIS
A SOLUTION TO PROBLEM 9 ON HW 6

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Here is a solution of Problem 9 on Homework 6.

Solution: Write down the statement that 0 is not a limit of the sequence $\{na_n\}$:

(1) there is an $\epsilon > 0$: for each $N \geq 1$ there is an $n \geq N : na_n \geq \epsilon$.

We want to find a contradiction with this assumption. Of course, the contradiction will come from the fact that the series converges,

Let us get back to the statement about 0 not being a limit. We can find n_1 such that $n_1 a_{n_1} \geq \epsilon$, then choose $n_2 \geq n_1 + 1$ such that $n_2 a_{n_2} \geq \epsilon$, then choose $n_3 \geq n_2 + 1$ such that $n_3 a_{n_3} \geq \epsilon$, and so on. This way we will get a subsequence of $\{a_{n_k}\}$ such that $n_k a_{n_k} \geq \epsilon$. Then the nth partial sum s_{n_k} of the series $\sum a_n$ can be estimated as follows:

$$\begin{aligned} s_{n_k} &= a_1 + \cdots + a_{n_1} + a_{n_1+1} + \cdots + a_{n_2} + \cdots + a_{n_k} \\ &\geq n_1 a_{n_1} + (n_2 - n_1) a_{n_2} + \cdots + (n_k - n_{k-1}) a_{n_k} \\ &\geq \epsilon\left(\frac{n_1}{n_1} + \frac{n_2 - n_1}{n_2} + \cdots + \frac{n_k - n_{k-1}}{n_k}\right). \end{aligned}$$

Here in the first inequality we used the fact that $a_n \geq a_{n+1}$ for all n, and the second inequality came from the assumption about 0 not being a limit of $\{a_n\}$. We need to estimate the result so as to see that the series diverges.

Digression: One estimate can be done like that: since $n_1 < n_2 < \cdots < n_k$, we have

$$s_{n_k} \geq \epsilon\left(\frac{n_1}{n_k} + \frac{n_2 - n_1}{n_k} + \cdots + \frac{n_k - n_{k-1}}{n_k}\right) = \epsilon \frac{n_k}{n_k} = \epsilon.$$

Unfortunately, this does not contradict the convergence of the series. Therefore, we need to look for a finer estimate.

Let us continue with

(2)
$$s_{n_k} \geq \epsilon\left(\frac{n_1}{n_1} + \frac{n_2 - n_1}{n_2} + \cdots + \frac{n_k - n_{k-1}}{n_k}\right) = \epsilon\left(1 + \left(1 - \frac{n_1}{n_2}\right) + \cdots + \left(1 - \frac{n_{k-1}}{n_k}\right)\right).$$

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Digression: If we had (1+ something) instead of (1− something), we would have $s_{n_k} \geq \epsilon k$, which would grow unboundedly. Unfortunately with the minuses, it does not work. We need to fine tune the estimate for s_{n_k}.

Note that we can use the fact that in (1) for any $N \geq 1$, there exists an n, in a better way. Namely, each time choosing n_{i+1} using (1), we can make sure it is at least $2n_i$.

Digression: How can you arrive at this choice? You stare at the estimate (2) and try to see, if there might be a reason why the last sum grows unboundedly. There is no immediate reason: for instance, all the n_{i-1}'s can be as close to their respective n_i's as being adjacent, in which case n_{i-1}/n_i will be close to 1, and the ith term in the sum will be close to $1/n_i$, *i.e.*, too small. However, in this case, that is to say, if the n_i's happened to be successive naturals, the partial sum will be a partial sum of the harmonic series $\sum 1/n$, which also diverges, but then there is a question about how to deal with intermediate situations, when the n_i's are not too close to each other and not too far. Thus, it is better to try to see if fiddling with the choice of n_i's, one can make each term in (2) large enough. How large? No way to make them larger then 1. On the other hand, 0 is not large enough. What is the most natural choice then? One half.

So, choosing $n_{i+1} \geq 2n_i$ as per (1), we get

$$1 - \frac{n_i}{n_{i+1}} \geq 1 - \frac{n_i}{2n_i} = \frac{1}{2},$$

and

$$s_{n_k} > \frac{\epsilon k}{2}.$$

This implies that the sequence of partial sums diverges and so does the series.