MATH 5615H: HONORS ANALYSIS
SAMPLE FINAL EXAM (PART II)

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You may not use a calculator, notes, books, etc. Only the exam paper, scratch paper, and a pencil or pen may be kept on your desk during the test. You must show all work.

Good luck!

Problem 1. Suppose that $f \in C^{n+1}(I)$, *i.e.*, has continuous derivatives through order $n+1$ for some open interval I with $0 \in I$ and some $n \geq 1$. Suppose also that there is a polynomial $P(x)$ of degree $\leq n$ such that

$$|f(x) - P(x)| \leq |x|^{n+1}.$$

Prove that the polynomial $P(x)$ is the Taylor polynomial centered at 0. That is, prove that

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

Problem 2. Suppose $f \in \mathcal{R}_0^1$. Prove that $\lim_{\varepsilon \to 0+} \int_{[0,\varepsilon]} f(x) dx = 0$.

Problem 3. Let c be a point on the closed interval $[a, b]$. Assume that $\{x_n\} \subseteq [a, b]$ is a sequence in $[a, b]$ such that every convergent subsequence of $\{x_n\}$ converges to c. Prove that the sequence $\{x_n\}$ converges.

Problem 4. Let $\{c_n\}$ be any sequence of positive numbers. Prove that

$$\liminf_n \frac{c_{n+1}}{c_n} \leq \liminf_n \sqrt[n]{c_n}.$$

Problem 5. Let $f : (0, 1) \to \mathbb{R}$ be a continuous function. Assume also that $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 1-} f(x)$ exist and are finite. Prove that $f(x)$ is bounded on $(0, 1)$.

Problem 6. Compute

$$\int_0^1 x^m (1-x)^n dx.$$

Problem 7. Suppose $\{a_n\}$ is a sequence of positive numbers. Let $s_n = a_1 + \cdots + a_n$ be the nth partial sum of the corresponding series. Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

Use this to show that the series $\sum \frac{a_n}{s_n^2}$ converges.

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