Math 5616H **Homework 10**

Posted: 4/11; Updated 4/15; Due: Friday, 4/17/2015

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 11: Sections 8, 10-11, 13-18.

Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a *homeomorphism*, *i.e.*, a continuous map $\mathbb{R}^n \to \mathbb{R}^n$ which has a continuous inverse. Suppose that f takes sets of Lebesgue measure zero to sets of Lebesgue measure zero. Prove that f takes Lebesgue measurable sets to Lebesgue measurable sets.

Problem 2. If $(X, \mathfrak{M}, \mu)$ is a measure space, $f : X \to [-\infty, \infty]$ is measurable, and $B \subset \mathbb{R}$ is a Borel set, show that the preimage $f^{-1}(B)$ is measurable in X. (I recall that a subset A in a measure space $(X, \mathfrak{M}, \mu)$ is called *measurable*, if $A \in \mathfrak{M}$. Sets measurable with respect to an outer measure is a different notion, even though the two notions become the same, when the measure is constructed out of an outer measure. In this problem, we are talking about an arbitrary measure.)

Problem 3. Suppose $f : X \to [-\infty, \infty]$ is a function on a measure space. Show that f is measurable iff the preimage $f^{-1}(a, \infty]$ is measurable for all $a \in \mathbb{Q}$.

Problem 4. Suppose X is a measure space, $f : X \to \mathbb{R}$ is measurable, and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Show that $g \circ f : X \to \mathbb{R}$ is measurable.

Problem 5. Describe the conditions on a subset $A \subset X$ under which the characteristic function $\chi_A : X \to \mathbb{R}$ of A, defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

is measurable.

Problem 6. Suppose $(f_i)_{i \in \mathbb{N}}$ is a sequence of measurable functions on a *complete* measure space X, which means that any subset of a measure-zero set in X is measurable. Suppose also that the sequence converges pointwise *almost everywhere* to a function f, *i.e.*, for every $x \in X$, except for members of a set a of measure zero. Show that f is measurable. You can use 11.17.

Problem 7. Show that Lebesgue measure is *translation invariant*, that is,

$m(A+x) = m(A)$ for every Lebesgue measurable $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$,

where $A + x := \{a + x \mid a \in A\}$.