Math 5616H **Homework 3**

Posted: 2/7; Due: Friday, 2/13/2015

The problem set is due at the beginning of the class on Friday.

Reading: 7.16-17 and 7.26-33.

Problem 1. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and

$$\int_a^b x^n f(x)dx = 0$$

for all integers $n \geq 0$. These integrals are called the *moments of* f. The conclusion addresses the question of uniqueness in a *moment problem.*

(1) Evaluate $\int_a^b P(x)f(x)dx$ for any polynomial $P(x)$.
(2) Prove that $\int_a^b (f(x))^2 dx = 0$.
(3) Show that $f(x) = 0$ for all $x \in [a, b]$.

Problem 2. Which of the following sequences $\{f_n\}$ of functions converges uniformly on $[0, 1]$?

(1) $f_n(x) = nx^2(1 - x)^n$;
(2) $f_n(x) = n^2x(1 - x^2)^n$;
(3) $f_n(x) = n^2x^3e^{-nx^2}$;
(4) $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$.

Problem 3. Let $A = \{f \in \mathcal{C}([0, 1], \mathbb{R}) \mid |f(x)| \leq 1 \text{ for all } x \in [0, 1]\}$.

(1) Show that A is a closed, bounded subset in $\mathcal{C}([0, 1], \mathbb{R})$.
(2) Show that the sequence $f_n(x) = x^n$ in A does not have a convergent subsequence.
(3) Explain why the fact that A is closed and bounded (Part (1)) does not contradict the fact that it is not compact, which follows from Part (2).

Problem 4. Let A be the same as in the preceding problem. Let $U_n := \{f \in \mathcal{C}([0, 1], \mathbb{R}) \mid |f(0) - f(1/n)| < 1\}$ for $n \in \mathbb{N}$. Show that $\{U_n\}$ is an open cover of A but does not admit a finite subcover. (This is another blow to the Heine-Borel principle!)

Problem 5. A metric space is called *separable* if it has a countable dense subset. For instance, $\mathbb{R}$ is separable, because $\mathbb{Q}$ is countable and dense.

(1) Prove that the metric space $\mathbb{C}$ of complex numbers is separable.
(2) Prove that $\mathcal{C}([0, 1], \mathbb{R})$ is separable.

Problem 6. Consider the set of polynomials $P(x)$ that have only terms of even degree, such as $x^6 - 3x^2 + 7$, but not $x + 2$ or $2x^4 - x^3 + 4$.

(1) Prove that these polynomials are dense in $\mathcal{C}([0, 1], \mathbb{R})$.
(2) Is this true for $\mathcal{C}([-1, 1], \mathbb{R})$?