Math 5616H **Homework 4**

Posted: 2/14; Updated: 02/19; Due: Friday, 2/20/2015

The problem set is due at the beginning of the class on Friday.

Reading: 7.3-6, 7.16-25.

Problem 1. Define a sequence of polynomials $P_0, P_1, P_2, \dots$ by $P_0(x) = 0$ and

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - (P_n(x))^2}{2}, \qquad n = 0, 1, 2, \dots$$

Prove that the sequence converges uniformly to the function $|x|$ on the interval $[-1, 1]$. *Hint*: Note that all P_n's are even and consider the interval $[0, 1]$. Then apply Dini's theorem (7.13).

Problem 2. Consider the following functional series:

$$e^{-x} + 2e^{-2x} + \dots + ne^{-nx} + \dots,$$

which we understand as a sequence of partial sums. Show that the series converges pointwise on $(0, +\infty)$ to a continuous function $f(x)$. Compute $\int_{\log 2}^{\log 3} f(x)dx$. *Hint*: For continuity use the material on power series we studied last term. For computation of the integral, use Dini's theorem (7.13) and the theorem on passing to limit in an integral.

Problem 3. Prove that the following functional series:

$$\frac{\sin 2\pi x}{2} + \frac{\sin 4\pi x}{4} + \dots + \frac{\sin 2^n \pi x}{2^n} + \dots$$

converges uniformly on $(-\infty, +\infty)$. Show that it cannot be differentiated termwise on any interval $[a, b]$.

Problem 4. Show that

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \dots + \frac{mx^{m-1}}{1+x^m} + \dots = \frac{1}{1-x},$$

where $m = 2^{n-1}$ and $-1 < x < 1$, in the sense of pointwise convergence.

Problem 5. Suppose $\mathcal{F} = \{f_i \mid i \in I\}$ be a finite collection of real- or complex-valued functions on a metric space X, *i.e.*, the set I is finite. Show that $\mathcal{F}$ is equicontinuous if and only if all the f_i's are uniformly continuous.

Problem 6. Consider the infinite sequence of functions

$$f_n(x) = \frac{x}{x + \frac{1}{n}}, \qquad x \in [0, 1], n \in \mathbb{N}.$$

Show that each function f_n is uniformly continuous, but the collection is not equicontinuous by showing that it has no uniformly convergent subsequence and applying the Arzelà-Ascoli theorem (7.25).

Problem 7. Consider and infinite collection $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ of functions

$$f_n(x) = e^{-n(x-n)^2}, \qquad x \in \mathbb{R}.$$

(1) Show that $\mathcal{F}$ is closed in $\mathcal{BC}(\mathbb{R}, \mathbb{R})$ but not closed in $\mathcal{C}([-a, a], \mathbb{R})$ for any $a > 0$. *Hint*: To show closedness over $\mathbb{R}$, estimate the distance between f_n and f_m by a constant, say, 1/2. This would imply that this set is discrete, i.e., each point of it being open, thereby making all subsets open and closed.

(2) Show that this collection is not equicontinuous on $\mathbb{R}$ but is equicontinuous on an interval $[-a, a]$ for any $a > 0$.