Math 5616H **Homework 6**

Posted: 3/8; Updated: 3/20; Due: Monday, 3/23/2015

The problem set is due at the beginning of the class on Monday.

Reading: 9.15-18, 20-21, 39, 41.

Problem 1. Let $U \subset \mathbb{R}^n$ be an open set. Show that if $f_k : U \to \mathbb{R}$, $k = 1, \ldots, m$ are differentiable, then $f = (f_1, \ldots, f_m)$ is differentiable on U and $Df = (Df_1, \ldots, Df_m)$.

Problem 2. Show that the maximum value of the directional derivative of $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, at a point x is the direction of the gradient $\nabla f(x)$ and the value of this derivative is $||\nabla f(x)||$. *Hint*: You can use the relation between the dot product of two vectors, their norms, and the angle between them.

Problem 3. Let

$$f(x,y) = \begin{cases} 0 & \text{when } (x,y) = (0,0), \\ \frac{xy}{x^2+y^2} & \text{otherwise.} \end{cases}$$

Show that $D_1 f(0,0) = D_2 f(0,0) = 0$, but nonetheless f is not continuous at the origin, and hence not differentiable there.

Problem 4. Let

$$f(x,y) = \begin{cases} 0 & \text{when } (x,y) = (0,0), \\ \frac{x^3}{x^2+y^2} & \text{otherwise.} \end{cases}$$

Show that f has a directional derivative in every direction at the origin, but f is not differentiable at the origin. *Hint*: Show that the formula $D_v f(x) = \nabla f(x) \cdot v$ for $||v|| = 1$ (Eqn. (40) in Chapter 9) fails.

Problem 5. Let

$$f(x,y) = \sqrt{|x| + |y|}.$$

Find those points in $\mathbb{R}^2$ at which f is differentiable. *Hint*: If near a point (x,y), say, in the second quadrant, the function is equal to $\sqrt{-x+y}$, then it will be the composition of two differentiable functions, thereby, differentiable by the "Chain Rule."

Problem 6. Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x,y) = x \cdot y.$$

Show that f is differentiable on $\mathbb{R}^n \times \mathbb{R}^n$ and that $Df(a,b)(x,y) = b \cdot x + a \cdot y$.

Problem 7 (Cauchy-Riemann Equations)**.** Let $U \subset \mathbb{C}$ be an open set. A function $f : U \to \mathbb{C}$ is called *complex differentiable* at $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. A function f is called *analytic* on U if f is complex differentiable at each point of U. Let $f(z) = u(x, y) + iv(x, y)$, where u and v are functions $U \to \mathbb{R}$ and z is written in the form $z = x + iy$ with $x, y \in \mathbb{R}$. Show that if f is analytic on U, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at all points of U.

Problem 8 (Mean Value Theorem)**.** Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ be differentiable on U. Let x, y be two distinct points in U such that the line segment joining them lies entirely in U. Show that there exists $\xi \in (0, 1)$ such that

$$f(y) - f(x) = Df(z)(y - x),$$

where $z = (1 - \xi)x + \xi y$.

Problem 9. Deduce from the previous problem that if $Df(x) = 0$ for all $x \in U$, then f is constant on U, provided U is connected. *Hint*: This problem does not obviously follow from the previous one, because a connected set is not necessarily convex. To use connectedness, pick a point $x \in U$ and show that the set of points $y \in U$ at which $f(y) = f(x)$ is open.

Problem 10. Let

$$f(x, y) = \begin{cases} 0 & \text{when } (x, y) = (0, 0), \\ \frac{x^3y - xy^3}{x^2+y^2} & \text{otherwise.} \end{cases}$$

Show that f is differentiable everywhere. Show that $D_{12}f(0, 0)$ and $D_{21}f(0, 0)$ exist, but $D_{12}f(0, 0) \neq D_{21}f(0, 0)$.