Math 8202 **Homework 4**

Date due: February 22, 2010. There will be a quiz on this day.

Hand in only the starred questions. One of the other problems may show up on the quiz.

If you want me to do in class any of the questions I have given you, do let me know (on or after the due date for the assignment). For instance, if you want me to do on February 22 or later any of the questions below, please tell me.

Section 4.5: 8, 10, 11, 14, 15, 16, 17*, 18, 21, 24*, 30, 32, 33*, 34, 35, 41, 42* (I list a lot of questions, and I expect that it will be appropriate for you to skim over many of them, simply looking to make sure you can do them.)

There are many questions on Sylow's theorem in the preliminary written exams from past years. For example, the question which appeared as question 5 in the Spring of 1994 is challenging:

J. (Spring 1994)

(1) Let G be a finite group. Prove that the number of conjugates in G of a subgroup H equals the index of its normalizer $N_G(H)$ in G.

(2) Let now G be a simple group of order $1092 = 3 \cdot 4 \cdot 7 \cdot 13$.

  (a) Find the number of Sylow 13-subgroups and the number of Sylow 7-subgroups of G.

  (b) Prove that G has a single conjugacy class of subgroups of index 14.

  (c) Prove that G has no subgroup of index 13.

[You may assume Sylow's theorems.]

K*. Let G be a finite group and let $O_p(G)$ denote the unique largest normal p-subgroup of G which contains all other normal p-subgroups of G. Establish its existence and show that $O_p(G)$ equals the intersection of all the Sylow p-subgroups of G:

$$O_p(G) = \bigcap_{P \text{ a Sylow } p\text{-subgroup}} P.$$

[You do not have to establish the existence specifically. Show that the group on the right contains every normal p-subgroup of G and is a normal p-subgroup of G.]

L. Let G be a finite group and H a subgroup. Let P_H be a Sylow p-subgroup of H. Prove that there exists a Sylow p-subgroup P of G such that $P_H = P \cap H$.

M. Prove that one of the Sylow subgroups of a group of order 40 is normal. Give an explicit example of a group of order 40 in which the Sylow 2-subgroup is not normal.

Section 4.6: 2*, 3, 4, 5, 6