MATH 8211: COMMUTATIVE AND HOMOLOGICAL ALGEBRA PROBLEM SET 1 (PART I), DUE OCTOBER 1, 2003

SASHA VORONOV

This is Part I of the problem set.

I encourage you to cooperate with each other on the homeworks.

Reminder: all rings are commutative with an identity element 1, all ring homomorphisms carry 1 to 1, and a subring shares the same identity element with the ring.

Problem 1. Let A be a UFD. A polynomial

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \in A[X]$$

is *primitive* if its coefficients a_i have no common factors in A (other than units). Prove *Gauss' Lemma*: the product of two primitive polynomials is primitive.

Problem 2 (This problem is not for credit. You may do it for fun)**.** Prove the cases $n = 3$ and $n = 4$ of Fermat's Last Theorem. [Harder, a hint will be given later. So far: use the discussion of $\mathbb{Z}[\sqrt[n]{1}]$ in the first lecture].

Problem 3. Let $\phi : A \to B$ be a ring homomorphism. Prove that ϕ^{-1} takes prime ideals of B to prime ideals of A. [In particular, if $A \subset B$ and P is a prime ideal of B, then $A \cap P$ is a prime ideal of A].

Problem 4. Prove or give a counterexample:

(1) the intersection of two prime ideals is prime;
(2) the ideal $P_1 + P_2$ generated by two prime ideals P_1, P_2 is again prime;
(3) if $\phi : A \to B$ is a ring homomorphism, then ϕ^{-1} takes maximal ideals of B to maximal ideals of A;
(4) the map ϕ^{-1} for a quotient homomorphism $\phi : A \to A/I$ takes maximal ideals of A/I to maximal ideals of A.

Problem 5.
(1) If a is a unit and x is nilpotent, prove that $a + x$ is again a unit.
(2) Let A be a ring, and $I \subset \operatorname{nilrad} A$ an ideal; if $x \in A$ maps to an invertible element of A/I, prove that x is invertible in A.

Problem 6. Show that if A is a reduced ring and has finitely many minimal prime ideals P_i, i.e., minimal elements in the set of prime ideals of A, then $A \hookrightarrow \bigoplus_{i=1}^n A/P_i$; moreover the image has nonzero intersection with each summand.

Problem 7. Describe $\operatorname{Spec} \mathbb{R}[X]$ in terms of $\mathbb{C}$.

Problem 8. Let A be a ring with zerodivisors, i.e., not an integral domain. Prove that A has either nonzero nilpotent elements, or more than one minimal prime ideal.

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