MATH 8211: COMMUTATIVE AND HOMOLOGICAL ALGEBRA PROBLEM SET 2, DUE NOVEMBER 3, 2003

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I encourage you to cooperate with each other on the homeworks.

Convention: all rings are commutative with an identity element $1 \neq 0$, all ring homomorphisms carry 1 to 1, and a subring shares the same identity element with the ring.

Problem 1. For a ring A, prove that A^m and A^n are isomorphic as A-modules, if and only if $m = n$. [Hint: use the existence of maximal ideals.]

Problem 2. If A is a ring and I a finitely generated ideal which is *idempotent*, i.e., satisfies $I = I^2$, prove that I is generated by a single idempotent element. [Hint: use a corollary from the determinant trick we used to prove Nakayama's lemma.]

Problem 3. Let A be an Artinian integral domain (i.e., one whose ideals satisfy the descending chain condition). Prove that A is a field. Deduce that every prime ideal of an Artinian ring is maximal.

Problem 4. Prove the Hilbert basis theorem for the formal power series ring $A[[X]]$ for Noetherian A.

Problem 5. Exercise 13.2 of [E].

Problem 6. Exercise 13.3 of [E]. [An *affine ring* is a just finitely generated algebra over a field. Note also that the ring of invariants will automatically be Noetherian by the Hilbert basis theorem.]

Problem 7. Let $n \in \mathbb{Z}$ be a number not divisible by any p^3. Find the normalization (i.e., integral closure) of $\mathbb{Z}[\sqrt[3]{n}]$. [Hint: suppose $n = l^2 m$; then the field $\mathbb{Q}(\sqrt[3]{n})$ also contains $\sqrt[3]{lm^2}$. Write any element of $\mathbb{Q}(\sqrt[3]{n})$ in the form $a + b\sqrt[3]{n} + c\sqrt[3]{lm^2}$ with $a, b, c, \in \mathbb{Q}$ and calculate its minimal polynomial over $\mathbb{Q}$.]

Problem 8. Prove the following refinement of the *Noether normalization lemma.* Let A be a finitely generated algebra over an *infinite* field k. Then there exist elements $z_1, \dots, z_m \in A$ such that

(1) $z_1, \dots, z_m$ are algebraically independent over k;
(2) A is finite over $B = k[z_1, \dots, z_m]$; and
(3) $z_1, \dots, z_m$ are linear combinations of the generators of A.

Problem 9. How does the result about a bijection between k^n and m-Spec $k[X_1, \dots, X_n]$ for $k = \bar{k}$ follow from Exercise 4.27 of [E]?

Problem 10 (A version of Weak Nullstellensatz over an arbitrary field)**.** Let k be a field. For an ideal $J \subset k[X_1, \dots, X_n]$ and an extension field $k \subset K$, define a *K-valued point* of $V(J)$ to be a point $(a_1, \dots, a_n) \in K^n$ such that $f(a_1, \dots, a_n) = 0$ for all $f \in J$. State an prove a version of the weak Nullstellensatz (on the structure of maximal ideals of $A = k[X_1, \dots, X_n]/J$) in terms of K-valued points of $V(J)$ for all algebraic extension fields K of k.

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Problem 11. Let k be a field and $k \subset K$ a Galois field extension with Galois group $G = \mathrm{Gal}(K/k)$. Prove that two K-valued points $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ of $V(J)$ correspond to the same maximal ideal of $k[X_1, \dots, X_n]$, if and only if there is an element $\sigma \in G$ such that $(a_1, \dots, a_n) = (\sigma(b_1), \dots, \sigma(b_n))$. [Hint: how would you do this, if $n = 1$?]

Problem 12. Exercise 4.11.a of [E].

Problem 13. Exercise 4.33 of [E].

Problem 14. Show that the Nullstellensatz implies

$$\operatorname{rad} J = \bigcap_{\substack{m \in \text{m-Spec}\, A \\ m \supset J}} m$$

for any ideal $J \subset A = k[X_1, \dots, X_n]$, when $k = \bar{k}$.

Problem 15. Let A and B be geometric rings over an algebraically closed field k, *i.e.*, finitely generated, reduced k-algebras, $\phi : A \to B$ a k-algebra homomorphism, and

$$\phi^\sharp : \text{m-Spec}\, B \to \text{m-Spec}\, A$$

the inverse-image map $\phi^\sharp(m) := \phi^{-1}(m)$. Describe $\phi^\sharp$ as a polynomial map between the varieties $\text{m-Spec}\, B$ and $\text{m-Spec}\, A$ corresponding to A and B. [A polynomial map is defined in coordinates by polynomials.]