Math 8253 **Homework 2**

Posted: 10/05/2017; 10/09: Corrected #2 and changed due date; Due: Monday, 10/16

The problem set is due at the beginning of the class on Monday, October 16.

Reading: **Class notes**. **Vakil**: Sections 3.2.9-10, 2.1, 2.2, 3.2.9-10, 4.1 through Exercise 4.1.B. **Hartshorne**: Sections II.1, II.2 before graded rings.

Problem 1. Let A be a ring and $S \subset A$ a multiplicative system. Prove that the localization homomorphism $\tau : A \to A_S$ induces a homeomorphism

$$\tau^* : \operatorname{Spec} A_S \xrightarrow{\sim} \bigcap_{f \in S} D(f) \subset \operatorname{Spec} A.$$

Problem 2. Let $\phi : A \to A'$ be a ring homomorphism and $\phi^* : \operatorname{Spec} A' \to \operatorname{Spec} A$ the corresponding map of spectra. Suppose $\operatorname{Spec} A'$ is irreducible and let ξ be its generic point: $\overline{\{\xi\}} = \operatorname{Spec} A'$. Show that the closure $\overline{\phi^*(\operatorname{Spec} A')}$ is irreducible and that $\phi^*(\xi)$ is a generic point of it.

Problem 3. Let R be a principal ideal domain (PID). Consider the natural inclusion $R \hookrightarrow R[t]$ and the induced map $\pi : \mathbb{A}^1_R := \operatorname{Spec} R[t] \to \operatorname{Spec} R$. ($\mathbb{A}^1_R$ is called the *affine line over* R.) For a point $s \in \operatorname{Spec} R$ show that the fiber $\pi^{-1}(s)$ is homeomorphic to the affine line $\mathbb{A}^1_{k(s)}$, where $k(s)$ is the residue field of s.

Problem 4. Let A be a ring and $X = \operatorname{Spec} A$. For each $f \in A$ set

$$S(f) := \{g \in A \mid D(f) \subset D(g)\}.$$

(1) Show that $S(f)$ is a multiplicative system and $f^n \in S(f)$ for all $n \geq 1$.
(2) Show that the resulting canonical localization homomorphism $A_f \to A_{S(f)}$ is actually an isomorphism.
(3) Define a contravariant functor $\mathcal{O}_X : \mathbb{D}(X) \to \mathbf{Ring}$ by

$$D(f) \mapsto A_{S(f)},$$
$$D(f) \subset D(g) \mapsto A_{S(g)} \to A_{S(f)},$$

where the last homomorphism is due to the fact that $S(g) \subset S(f)$.
(4) Show that the composition $\mathcal{O}_X \circ F$ with the forgetful functor $F : \mathbb{D}^\#(X) \to \mathbb{D}(X)$, $f \mapsto D(f)$, is naturally equivalent to $\mathcal{O}_X^\# : \mathbb{D}^\#(X) \to \mathbf{Ring}$, $f \mapsto A_f$.

Hint: See Vakil 4.1.1 for more details.

Problem 5. Let A be a ring and $X = \operatorname{Spec} A$. Given $f \in A$, take the localization A_f and its spectrum $X_f = \operatorname{Spec} A_f$. Consider the homeomorphism $\tau^* : X_f \to D(f) \subset X$ of Problem 1.

(1) Show that the collection of sets $D(g) \subset X$ such that $D(g) \subset D(f)$ is in bijection with the collection of basic open sets $D_f(h)$ in X_f.
(2) The restriction of the functor $\mathcal{O}_X : \mathbb{D}(X) \to \mathbf{Ring}$ to the subcategory of subsets $D(g) \subset D(f)$ is equivalent to the functor $\mathcal{O}_{X_f} : \mathbb{D}(X_f) \to \mathbf{Ring}$. You may assume that $\mathcal{O}_X(D(f)) = A_f$ in view of the canonical isomorphism $A_f \to A_{S(f)}$ of Problem 4(2).

Problem 6. Give an example of a presheaf $\mathcal{F}$ on a topological space such that $\mathcal{F}$ satisfies the locality axiom of a sheaf but not the gluing axiom. Similarly, give an example of a presheaf that satisfies the gluing axiom but not the locality axiom.