Math 8253 **Homework 3**
Posted: 10/22/2017; Problem 5 corrected 10/23; Due: Monday, 10/30

The problem set is due at the beginning of the class on Monday, October 30.

Reading: **Class notes**. **Vakil**: Sections 4.1 after Exercise 4.1.B, 1.4, 2.7, 2.2.3-6, 2.2.11, 2.4, 2.3.A, 2.2.13, 4.3, 6.2-3 through 6.3.D. **Hartshorne**: Sections II.2 (pp. 70-72), II.5, skipping graded rings and Proj.

Problem 1. Describe the structure sheaf $\mathcal{O}_X$ for $X = \operatorname{Spec} \mathbb{Z}$ by exhibiting the restriction morphisms for all open sets. Recall that open sets of X are $\emptyset$, X, and the complements to sets consisting of finitely many closed points p, corresponding to prime numbers.

Problem 2. Suppose that $(G_i, f_{ij})_{i,j \in I}$ is an inductive system of groups indexed by a directed set I, *i.e.*, a functor from I regarded as a category with morphisms $i \le j$ to the category of groups. Forget the group structure for the time being and consider $(G), f_{ij})$ as an inductive system of sets. Take the standard construction of an inductive limit as the set of equivalence classes on the disjoint union of the G_i's,

$$G = \coprod_{i \in I} G_i / \sim,$$

where for $x \in G_i$ and $y \in G_j$, $x \sim y$ if $f_{ik}(x) = f_{jk}(y)$ for some k such that $i, j \le k$. Now recall that (G_i, f_{ij}) was an inductive systems of groups. Equip the set G with the structure of a group and in such a way that the canonical maps $f_i : G_i \to G$ are group homomorphisms and show that G with these homomorphisms possesses the universal property.

Problem 3. Let A be a integral domain. For any point x in $X = \operatorname{Spec} A$, let A_x be the localization at the corresponding prime ideal. View A_x as a subring of the field K of fractions of A. For any open set $U \subset X$, show that $\mathcal{O}(U)$ is canonically isomorphic to $\bigcap_{x \in U} A_x$.

Problem 4. Show that any ring A is canonically isomorphic to the projective limit of all its localizations A_f, $f \in A$.

Problem 5. Consider the affine scheme $X = \operatorname{Spec} \mathbb{Z}[t]$ and take its open subscheme U corresponding to the union of the basic open sets $D(2)$ and $D(t)$. Show that the subscheme U is not affine. *Hint*: Determine the ring $\mathcal{O}_X(U)$.

Problem 6. Let $\mathcal{O}(U)$ be the algebra of holomorphic functions on U, an open subset of the complex plane $\mathbb{C}$. Show this presheaf is a sheaf and that the stalk $\mathcal{O}_0$ at the origin $0 \in \mathbb{C}$ may be identified with convergent power series

$$\left\{ \sum_{n=0}^{\infty} a_n z^n \;\middle|\; \sum_{n=0}^{\infty} a_n z^n \text{ converges for some } z \in \mathbb{C} \setminus \{0\} \right\}.$$

Problem 7. Show that the sheafification map $\mathcal{F} \to \mathcal{F}^+$ for a presheaf $\mathcal{F}$ over a topological space X induces an isomorphism $\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_x^+$ on the stalks at each $x \in X$.

Problem 8. Exercise II.1.2 from Hartshorne.