MATH 8253: ALGEBRAIC GEOMETRY
HOMEWORK 4
UPDATED WITH HINTS ON 11/1
DUE WEDNESDAY, NOVEMBER 2, 2:17 P.M.

INSTRUCTOR: SASHA VORONOV

Review your course notes and read Gathmann's *Chapter* 6 : *Projective Varieties I: Topology* from *Section* 6.19 through the end and *Chapter* 7 : *Projective Varieties II: Ringed Spaces*. Do the following problem and Exercises 6.29, 6.31, 7.7, 7.8, 7.30.

Problem A: Prove that the line with a doubled origin is not separated (i.e., not a variety), see the discussion before Def. 5.17. *Hint*: Use the universal property of the product, Prop. 5.15.

Hints. 6.29: Work with vector subspaces of K^4 corresponding to all sorts of linear subspaces of $\mathbb{P}^3$ and use linear algebra.

6.31(b): Use 5.23(b).

7.7: Use 7.8, even you have not done it. I do not know how to do 7.7 without 7.8. I guess, Gathmann meant to write these problems in the opposite order.

7.8: The idea might be to see what this map is in an affine open chart $\mathbb{A}^m$, say, $x_0 \neq 0$, of the target space $\mathbb{P}^m$: the map is determined, at least in an open $U \subset \mathbb{P}^n$ by the pullbacks $g_1, \dots, g_m$ of the coordinate functions $x_1, \dots, x_m$ on $\mathbb{A}^m$. And those pullbacks must be regular on U, even quotients of homogeneous polynomials of same degree on, perhaps, some smaller open U. These determine where the points of U map to: they map to $[1 : g_1 : \cdots : g_m] \in \mathbb{A}^m \subset \mathbb{P}^m$. Try to get rid of the denominators and have no common factor of the resulting set of $m + 1$ homogeneous polynomials of same degree without affecting the map. Try to cover the source $\mathbb{P}^n$ with such U's. Show that on overlaps these $(m + 1)$-tuples of homogeneous polynomials are the same up to a constant factor.

Submit the homework by the start of our Wednesday, November 2, class meeting, *i.e.*, by 2:17 p.m. Please submit it electronically to Gradescope at
`https://www.gradescope.com/courses/445177`,
which you can access through Canvas.