Math 8254 **Homework 1**

Posted: 01/09/2018; Comment added to Hartshorne's problem on 01/24;
Due: Monday, 01/29/2018

The problem set is due at the beginning of the class on Monday, January 29.

Reading: **Class notes from last term**. (If you do not have them, ask someone for notes and use this as an excuse to make friends with a classmate and possibly form a study group.) **Vakil**: Sections 4.4.4–10, 9.1. **Hartshorne**: Exercise II.1.22, Sections II.2.3.5–6, Exercise II.2.12, Exercise II.5.17(c), Section II.3 from Definition on p. 87 through mid-page 89. (All the exercises in this part of the assignment are for reference only; you do not have to solve them, unless you want to.)

Problem 1. Given a scheme X, let $h_X : \mathbf{Sch} \to \mathbf{Set}$ be the associated *functor of points*, which maps a scheme S to the set $h_X(S) := \operatorname{Mor}(S, X)$ of morphisms $S \to X$ and is defined on morphisms in an obvious way.

(1) Show that a natural transformation $h_x \to h_{X'}$ between functors of points of schemes X and X' is induced by a morphism $X \to X'$ of schemes. Show the same works for isomorphisms.
(2) Let h_X^{aff} be the restriction of h_X to the full subcategory of affine schemes in **Sch**. Show that the assertion of (1) remains true for natural transformations $h_X^{\text{aff}} \to h_{X'}^{\text{aff}}$ between functors of points on the category of affine schemes, even if X and X' are not necessarily affine.

Problem 2. Exercise 4.4.F in Vakil's *The Rising Sea*, November 18, 2017, version.

Problem 3. For a base scheme S and nonnegative integers m, n construct two universal (in S) S-morphisms $\mathbb{P}_S^m \to \mathbb{P}_S^n$, where $\mathbb{P}_S^k := \mathbb{P}^k \times_{\operatorname{Spec} \mathbb{Z}} S$, such that for $S = \operatorname{Spec} K$, where K is a field, the resulting maps $\mathbb{P}_K^m(K) \to \mathbb{P}_K^n(K)$ on K-points are

(1) $[x_0 : \dots : x_m] \mapsto [x_0 : \dots : x_m : 0 : \dots : 0]$ for any $0 \le m \le n$;
(2) $[x_0 : \dots : x_m] \mapsto [M_0(x) : \dots : M_n(x)]$, where $M_0, \dots, M_n$ are the $n = \binom{m+d}{d}$ monomials in $x = (x_0, \dots, x_m)$ of degree d for any $m \ge 0$ and $d \ge 1$. This is known as the *Veronese map*.

Problem 4. Show that in general, unlike the relative affine n-space $\mathbb{A}_S^n$, the relative projective n-space $\mathbb{P}_S^n$ is not isomorphic to an n-fold fibered product of $\mathbb{P}_S^1$ with itself. *Hint*: Count $\mathbb{F}_p$-points in $\mathbb{P}_{\mathbb{F}_p}^2$ and $\mathbb{P}_{\mathbb{F}_p}^1 \times_{\mathbb{F}_p} \mathbb{P}_{\mathbb{F}_p}^1$.

Problem 5. Exercise 4.4.B in Vakil's *The Rising Sea*.

Problem 6. Exercise 4.4.E in Vakil's *The Rising Sea*.

Problem 7. Show that $X \times Y := X \times_{\operatorname{Spec} \mathbb{Z}} Y$ is a categorical product in the category of schemes. *Hint*: This is a purely categorical statement having a purely categorical proof.

Problem 8. Exercise II.3.9(b) in Hartshorne. Here $k(s)$ means the field of rational functions of one variable s, the field of fractions of the polynomial algebra $k[s]$.