Math 8254 **Homework 2**
Posted: 2/6/18; Problem 2 corrected 2/14/18; Due: Wednesday, 2/14/18

The problem set is due at the beginning of the class on Wednesday, February 14 (or in my mailbox).

Reading: **Class notes**. **Vakil** (11/18/17 version): Sections 9.2-3 through 9.3.3, 3.6.12-S, 10.1 through 10.1.10, and 11.1 through 11.1.2. **Hartshorne**: Sections II.3 (pp. 83-84, 86, 89-90), II.4 (pp. 95-100), I.1 (pp. 5-7).

Problem 1. Let $X = \operatorname{Spec} \mathbb{Z}[x, y]/(y^2 - 2x)$ be the curve in $\mathbb{A}^1_{\mathbb{Z}}$ given by the equation $y^2 - 2x = 0$, a parabola. X is a $\mathbb{Z}$-scheme. If we change the base to $\mathbb{C}$, $X_{(\mathbb{C})} = X \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C}$, the curve will be isomorphic to $\mathbb{A}^1_{\mathbb{C}}$, because $\mathbb{C}[x, y]/(y^2 - 2x) \cong \mathbb{C}[y]$. Describe the fiber X_p of X over each point $p \in \operatorname{Spec} \mathbb{Z}$ as the spectrum of an algebra, such as we described $X_{(\mathbb{C})}$ as $\operatorname{Spec} \mathbb{C}[x, y]/(y^2 - 2x)$. For which p is X_p isomorphic to the affine line (over a suitable field)? Do not forget the generic point $\eta \in \operatorname{Spec} \mathbb{Z}$ corresponding to the zero ideal.

Problem 2. Show that a quasi-compact open embedding $U \subset X$ into an S-scheme X of finite type results in an S-scheme U of finite type.

Problem 3. Show that a scheme of finite type over a Noetherian scheme is Noetherian.

Problem 4. Prove that our definition of separatedness of an S-scheme X (the image $\Delta_X(X)$ of the diagonal morphism $\Delta_X : X \to X \times_S X$ is closed) is equivalent to the following. An S-scheme X is separated over S if and only if for any two S-morphisms $f, g : Y \to X$, the set $\{y \in Y \mid f(y) \equiv g(y)\}$ is closed in Y. Here $f(y) \equiv g(y)$ reads *f and g coincide scheme-theoretically at a point $y \in Y$* and means that the compositions $f \circ \iota_y$ and $g \circ \iota_y$ of f and g with $\iota_y : \operatorname{Spec} k(y) \to Y$ are equal. (This condition may be reworded as $f(y) = g(y) =: x$ and that the field homomorphisms $k(x) \to k(y)$ coming from the homomorphisms $f_y^\#, g_y^\# : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ of the stalks coincide.) *Hint*: Show that $f(y) \equiv g(y)$ if and only if $(f, g)(y) \in \Delta_X(X)$, where $(f, g) : Y \to X \times_S X$ is the canonical morphism induced by f and g.

Problem 5. Show that an A-scheme X is separated over a ring A, if and only if X is separated over the integers $\mathbb{Z}$.

Problem 6.
(1) Prove that if R is a ring with $\dim R = 0$, then $R = Q(R)$, the *ring of fractions of* R, *i.e.*, the localization with respect to the set of all non-zero divisors.
(2) Show that if R is *reduced*, *i.e.*, contains no nilpotents, has only a finite number of minimal prime ideals, and $R = Q(R)$, then $\dim R \leq 0$.

Problem 7. Show that for a point x in a scheme X, the codimension of the closed irreducible set $\overline{\{x\}}$ coincides with the Krull dimension of the stalk $\mathcal{O}_{X,x}$. The *codimension of a closed irreducible subset* $Y \subseteq X$ is the supremum of the lengths r of chains $Y \subset X_1 \subset \cdots \subset X_r$ of closed irreducible subsets of X that start with Y. Deduce that the dimension of a scheme is the supremum of the Krull dimensions of its stalks. *Hint*: See Vakil's Exercise 11.1.B. Note also that a nonempty open subset of an irreducible topological space is always dense in it.