Math 8254 **Homework 3**

Posted: 2/21/2018; Minor corrections: 2/22; Due: Wednesday, 2/28

The problem set is due at the beginning of the class on Wednesday, February 28.

Reading: **Class notes**. **Vakil** (11/18/17 version): Sections 3.6.12–13, Q, 11.1–2, 7.3.3–L, 8.1. **Hartshorne**: Sections II.3 (pp. 84–87), I.1 (pp. 5–7), Exercise II.5.17 (a–c).

Problem 1.
(1) Let K be a field and R be a K-algebra of finite type which is an integral domain (has no zero divisors, not be confused with being integral over K). Show that every irreducible closed subset of codimension 1 in $\operatorname{Spec} R$ has dimension $d-1$, where $d = \dim R$.
(2) Give an example of a Noetherian integral domain R such that the previous assertion does not hold for $\operatorname{Spec} R$. *Hint*: Try $R = K[[X]][Y]$, where $K[[X]]$ is the algebra of formal power series in X. It is Noetherian being a discrete valuation ring and thereby a principal ideal domain.

Problem 2. Let $q = p^n$ be a power of a prime p and X be an $\mathbb{F}_q$-scheme, where $\mathbb{F}_q$ is the Galois field of q elements. Since $\mathbb{F}_q$ is of characteristic p, X is automatically an $\mathbb{F}_p$-scheme and has a canonical Frobenius morphism $F : X \to X$. Then F^n, the n-fold iteration of F, is an affine $\mathbb{F}_q$-morphism (affine as a composition of such). Prove that the fixed points of F^n, *i.e.*, the set X^{F^n} of points $x \in X$ such that $F^n(x) \equiv \mathrm{id}_X(x)$ (scheme-theoretic coincidence of F^n and id_X at x from the previous homework) is exactly the set $X(\mathbb{F}_q) = \operatorname{Mor}(\operatorname{Spec} \mathbb{F}_q, X)$ of $\mathbb{F}_q$-points of X.

Problem 3. Exercise II.5.17(d) in Hartshorne. You may take II.5.7(a–c) for granted, as we have done all this in the past.

Problem 4. Prove that finite morphisms are stable under base change, *i.e.*, if $f : X \to S$ is a finite morphism and $g : T \to S$ is any morphism, then the canonical projection $T \times_S X \to T$ is finite. (In fact, a similar argument will work for affine morphisms or closed embeddings, but you do not have to do it.)

Problem 5. Show that, for a finite morphism $f : X \to Y$ of schemes, the preimage $f^{-1}(y)$ of a point $y \in Y$ is a finite set.

Problem 6.
(1) Show that, given a ring homomorphism $A \to B$ and a collection of elements $f_1, \dots, f_n \in A$ such that $\bigcup_{i=1}^n D(f_i) = \operatorname{Spec} A$ and the induced homomorphisms $A_{f_i} \to B_{f_i}$ are surjective for all $i = 1, \dots, n$, the homomorphism $A \to B$ is surjective.
(2) Show that for a morphism $f : X \to Y$ of schemes, the following two conditions for being a closed embedding are equivalent:
   (a) For any open affine $V \subseteq Y$, its preimage $f^{-1}(V)$ is affine and $f^{\#}(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ is surjective;
   (b) There exists an open affine cover $\{V_i \mid i \in I\}$ of Y such that for each $i \in I$ the preimage $f^{-1}(V_i)$ is affine and $f^{\#}(V_i)$ is surjective.

Problem 7. Show that a morphism $f : X \to Y$ of schemes is a closed embedding if and only if f is a homeomorphism of X onto a closed subset of Y and for every open affine subset V of Y, the ring homomorphism $f^{\#}(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ is surjective. *Hint*: Part of this is Vakil's Exercise 8.1.A, another part is an easy part of Exercise 8.1.K.

Problem 8. Prove that a morphism $f : X \to S$ is separated (with respect to either of the two notions we have used, see Homework 2) if and only if the diagonal morphism $\Delta_X : X \to X \times_S X$ is a closed embedding.