Math 8254 **Homework 4**
Posted: 3/4/2018; Due: Monday, 3/19

The problem set is due at the beginning of the class on Monday, March 19. **Reading assignment this time contains some important things not covered in class for studying on your own**.

Reading: **Class notes**. **Vakil** (11/18/17 version): Sections 4.5.4–11, 8.3.9–10, 10.1.5, 10.1.14, 6.4 through 6.4.D, 8.2.1–A, 8.2.4–G, 8.2.6, 8.2.8–K, 8.2.11–N. **Hartshorne**: Examples II.3.2.3 and 3.2.6, Section II.2 (pp. 76–77), Exercise II.2.14. (All the exercises in this part of the assignment are for reference only; you do not have to solve them, unless you are asked to below or want to.)

Problem 1. Vakil's Exercise 8.3.G.

Problem 2. Let $R = \bigoplus_{m \geq 0} R_m$ be a positively graded ring. Let $f \in R$ and $g \in R$ be homogeneous elements of positive degrees d and e, respectively, and $D_+(g) \subseteq D_+(f)$. As we know, this is equivalent to $\operatorname{rad}(g) \subseteq \operatorname{rad}(f)$ or $g^n = af$. Prove that in this case $(R_g)_0 = ((R_f)_0)_{g^d/f^e}$. (As always, the equality sign stands for a canonical isomorphism. Note also that $(R_g)_0$ is what Hartshorne denotes $R_{(g)}$, whereas $D_+(g)$ is what Vakil denotes $D(g)$.)

Problem 3. Let $R = \bigoplus_{m \in \mathbb{Z}} R_m$ be a graded ring and $f \in R_1$.

(1) Construct a ring homomorphism $\phi : (R_f)_0 \to R/(1-f)$.
(2) Show that $a \mapsto a/f^m$ for $a \in R_m$ defines a ring homomorphism $\psi : R/(1-f) \to (R_f)_0$ and that it is an inverse of ϕ.
(3) Now let $g \in R_d$, $d > 0$. Show that $(R_g)_0 = R^{[d]}/(1-g)$ and $R^{[d]} := \bigoplus_{m \geq 0} R_{dm} \subseteq R$.

Problem 4. Let R be a positively graded ring and $X = \operatorname{Proj} R$. Show that the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at $x \in X$ with $\mathfrak{p} := \mathfrak{p}_x \subset R$ the corresponding prime ideal is isomorphic to $(R_{[\mathfrak{p}]})_0$, where $R_{[\mathfrak{p}]} := S^{-1}R$, the localization of R with respect to the multiplicative system of all homogeneous elements in R not in $\mathfrak{p}$.

Problem 5. Vakil's Exercise 6.4.A.

Problem 6. Vakil's Exercise 6.4.B.

Problem 7. Vakil's Exercise 6.4.D.

Problem 8. Vakil's Exercise 8.2.B. *Hint*: It is a closed embedding with image $V_+(\operatorname{Ker} \phi)$. Remember that we use Hartshorne's notation V_+.