**Math 8254** **Homework 5**
**Posted: 3/23/2018; Added 8.2.11–N to reading: 03/26; Due: Monday, 4/2/2018**

The problem set is due at the beginning of the class on Monday, April 2.

**Reading**: **Class notes**. **Vakil** (11/18/17 version): Sections 10.1.7–E (We looked at algebraic $A$-schemes in class. A $k$-*variety* is just a reduced algebraic $k$-scheme.), 8.2.6–L, 9.6, 8.2.11–N, 9.3.F, 22.1, 22.3 through 22.3.2, 22.3.4, 22.4.1, 22.4.3. **Hartshorne**: Exercises I.3.14, I.2.12, I.2.14, I.3.4, I.3.16. II.5.11, Sections I.4 (pp. 28–29), II.7 (pp. 163–165). (All the exercises in this part of the assignment are for reference only; you do not have to solve them, unless you are asked to below or want to.)

**Problem 1.** Let $R = \bigoplus_{m\ge 0} R_m$ be a positively graded ring.

(1) Prove the equivalence of the following two conditions: (a) The irrelevant ideal $R_+$ is finitely generated; (b) The $R_0$-algebra $R$ is of finite type, *i.e.*, finitely generated as an $R_0$-algebra.
(2) Show that if $R$ is an $R_0$-algebra of finite type, then the $d$th Veronese subring $R^{[d]}$ is also an $R_0$-algebra of finite type and $R$ is a finite $R^{[d]}$-algebra, *i.e.*, $R$ is finitely generated as an $R^{[d]}$-module.

**Problem 2.** Show that a *projective algebraic $A$-scheme*, *i.e.*, a scheme isomorphic to $\operatorname{Proj} R$ for a positively graded algebra $R$ of finite type over $A = R_0$, is actually isomorphic to $\operatorname{Proj} S$, where $S$ is a standardly graded algebra of finite type over $S_0$. A grading is called *standard* if $S$ is generated by $S_1$ as an $S_0$-algebra. *Hint*: This is actually Vakil's Exercise 6.4.G, which also has a hint. Use the isomorphism $\operatorname{Proj} R \to \operatorname{Proj} R^{[d]}$ from the previous homework and prove the following statement. Under our assumptions about $R$, there exists a $d > 0$ such that $R^{[d]}$ is generated by $R_d$ over $R_0$ and thereby could be regraded to a standard grading. This statement may be proven by choosing a set of homogeneous generators $x_0, \dots, x_n$ of $R$ of various positive degrees $\gamma_0, \dots, \gamma_n$. Then show that for $m = \operatorname{LCM}(\gamma_0, \dots, \gamma_n)$, there is a multiple $d = cm$ of $m$ such that $R^{[d]}$ is generated by $R_d$.

**Problem 3.** Let $R = \bigoplus_{m\ge 0} R_m$ and $S = \bigoplus_{m\ge 0} S_m$ be positively graded rings with $A := R_0 = S_0$. Define the *Segre product* $R \#_A S := \bigoplus_{m\ge 0} R_m \otimes_A S_m$. Prove that $\operatorname{Proj}(R\#_A S) \cong \operatorname{Proj} R \times_A \operatorname{Proj} S$. *Hint*: This is actually Vakil's Exercise 9.6.D. Show that $(R_f)_0 \otimes_A (S_g)_0 \cong ((R\#_A S)_{f\otimes g})_0$ for homogeneous $f$ and $g$. Perhaps, I should discourage the categorical approach I suggested in class, because the best you can do this way would be to show that $\operatorname{Proj}(R\#_A S)$ is a product in the category of projective $A$-schemes $\operatorname{Proj} R$. You can still improve the situation by describing morphisms from arbitrary $A$-schemes to projective $A$-schemes $\operatorname{Proj} R$ in the spirit of how morphisms from schemes to affine schemes are described. But all that is a little project, while a direct argument is simple enough.

**Problem 4.** Prove that the fibered product (over $\operatorname{Spec} A$) of two projective algebraic $A$-schemes is a projective algebraic $A$-scheme.

**Problem 5.** Vakil's Exercise 8.2.I.

**Problem 6.** Vakil's Exercise 8.2.J.

**Problem 7.** Vakil's Exercise 9.6.C.

**Problem 8.** Vakil's Exercise 9.3.F.