**Math 8254** **Homework 6**
**Posted: 4/8/2018; Minor error in #1 corrected: 7/13/18 ; Due: Wednesday, 4/18/2018**

The problem set is due at the beginning of the class on Wednesday, April 18.

**Reading**: **Class notes**. **Vakil** (11/18/17 version): Sections 9.3.F, 22.1, 22.3 through 22.3.2, 22.3.4, 22.4.1, 22.4.3, 19.9.10 (p. 531; ignore the sheaves O(D)), 15.1-2, 15.3.1, 15.4 through 15.4.D, 15.4.2-3, 18.2 through 18.2.B. **Hartshorne**: Sections I.4 (pp. 28–29), II.7 (pp. 163–165), Example II.6.10.2 (p. 139; ignore the divisors), II.5 (pp. 116–119), III.4 through Lemma 4.1.

**Problem 1.** Let $A$ be a ring and $\mathfrak{a} \lhd A$ an ideal in it. Consider the corresponding closed affine subscheme $Z = \operatorname{Spec} A/\mathfrak{a}$ in $X = \operatorname{Spec} A$. Let $\operatorname{Bl}_Z(X) = \operatorname{Proj} A[\mathfrak{a}T]$, where $A[\mathfrak{a}T] = A \oplus \mathfrak{a}T \oplus \mathfrak{a}^2T^2 \oplus \dots$ is graded by $\deg T = 1$, be the blowup of $X$ along $Z$. The point of this problem to show that the *exceptional divisor*, *i.e.*, the scheme-theoretic preimage $E_Z(Z) := Z \times_X \operatorname{Bl}_Z(X)$ of $Z$ with respect to the canonical projection $\operatorname{Bl}_Z(X) \to X$ is indeed what is called an *effective Cartier divisor*, *i.e.*, a closed subscheme whose sheaf of ideals is locally generated by a non-zero divisor. An effective Cartier divisor will automatically be a subscheme of codimension one. We have seen in class that $E_Z(X) = \operatorname{Proj} A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T]$, where $A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T] = \bigoplus_{m \ge 0} (\mathfrak{a}^m/\mathfrak{a}^{m+1})T^m$. Divide the proof into the following three steps.

(1) For each $a \in \mathfrak{a}$, identify the algebra $(A[\mathfrak{a}T]_{aT})_0$ of global functions on the affine open subset $D_+(aT) = \operatorname{Spec}(A[\mathfrak{a}T]_{aT})_0$ of $\operatorname{Bl}_Z(X)$ as the subalgebra $A[a^{-1}\mathfrak{a}] = A + a^{-1}\mathfrak{a} + a^{-2}\mathfrak{a}^2 + \dots$ of $A_a$.
(2) Identify the quotient algebra $((A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T])_{aT})_0$ of global functions on the affine open subset $\operatorname{Spec}((A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T])_{aT})_0$ of $E_Z(X)$ as the quotient $A[a^{-1}\mathfrak{a}]/aA[a^{-1}\mathfrak{a}]$ of the algebra $A[a^{-1}\mathfrak{a}]$. This implies that the closed subscheme $E_Z(X) \subseteq \operatorname{Bl}_Z(X)$ is determined within the open subset $D_+(aT) \subseteq \operatorname{Proj} A[\mathfrak{a}T]$ by the principal ideal $(a) \lhd A[a^{-1}\mathfrak{a}]$.
(3) Conclude by observing that $a$ is a unit in the extension $A_a$ of $A[a^{-1}\mathfrak{a}]$ and that the affine open subsets $D_+(aT)$ cover $\operatorname{Bl}_Z(X)$.

**Problem 2.** Within the setup of the previous problem, show that the quasi-coherent sheaf $\mathcal{O}_{\operatorname{Bl}_Z(X)}(1)$ is isomorphic to the sheaf $\mathfrak{a}A[\mathfrak{a}T]^\sim$ associated with the graded ideal $\mathfrak{a}A[\mathfrak{a}T] \lhd A[\mathfrak{a}T]$ defining the exceptional divisor $E_Z(X)$.

**Problem 3.** Let $R = \bigoplus_{m \ge 0} R_m$ be a positively graded ring and $M = \bigoplus_{m \in \mathbb{Z}} M_m$ and $N = \bigoplus_{m \in \mathbb{Z}} N_m$ be two graded $R$-modules. Let $M \otimes_R N$ be their tensor product over $R$ with obvious grading.

(1) Construct a sheaf morphism $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to (M \otimes_R N)^\sim$. *Hint*: It is just enough to construct canonical homomorphisms $(M_f)_0 \otimes_{(R_f)_0} (N_f)_0 \to (M_f \otimes_{R_f} N_f)_0 = ((M \otimes_R N)_f)_0$.
(2) Show that the sheaf morphism from the previous part is an isomorphism if $R$ is standardly graded. In particular, $\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \xrightarrow{\sim} \widetilde{M(n)}$.
(3) Give an example of $R$ such that $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{O}_X(2)$ is not an isomorphism.

**Problem 4.** Let $\mathbb{P}^n_A = \operatorname{Proj} A[X_0, \dots, X_n]$ be the projective $n$-space over a ring $A$. We know from Homework 4 that given a homogeneous ideal $\mathfrak{a} \lhd A[X_0, \dots, X_n]$, the projection

$$A[X_0, \dots, X_n] \to A[X_0, \dots, X_n]/\mathfrak{a}$$

onto the quotient graded $A$-algebra $A[X_0, \dots, X_n]/\mathfrak{a}$ gives rise to a close embedding of $A$-schemes

$$\operatorname{Proj} A[X_0, \dots, X_n]/\mathfrak{a} \to \operatorname{Proj} A[X_0, \dots, X_n].$$

Prove that every closed embedding $Z \hookrightarrow \mathbb{P}^n_A$ of $A$-schemes is of this type. *Hint*: Use the Serre isomorphism $\Gamma_*(\mathcal{F})^\sim \to \mathcal{F}$. See more in Vakil 15.4.H.

**Problem 5.** Glue two copies of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[T]$ over a field $k$ via the $k$-algebra isomorphism $k[T, T^{-1}] \to k[T, T^{-1}]$ given by (1) $T \mapsto T$; (2) $T \mapsto T^{-1}$. The resulting $k$-scheme $X$ comes with an open cover $\mathcal{U}$ consisting of two affine lines and is (1) the affine line $\overline{\mathbb{A}}^1_k$ with a doubled point and, respectively, (2) the projective line $\mathbb{P}^1_k$. Compute the Čech cohomology groups $H^q(\mathcal{U}, \mathcal{O}_X)$, $q \geq 0$, in both cases.