Math 8254 **Homework 7**

Posted: 4/22/2018 (preliminary version); Problem 5 changed again: 05/15; Due: Wednesday, 5/9/2018

The problem set is due at the beginning of the class on Wednesday, May 9.

Reading: **Class notes**. **Vakil** (11/18/17 version): Sections 18.2 through 18.2.2, 18.2.4, 23.2, 23.4 through 23.4.3, 18.1.3, 18.3, 12.2 through 12.2.B, 14.3, 18.4 through 18.4.3, 18.5 through 18.5.4. **Hartshorne**: Sections III.1-2, III.4 after Lemma 4.1 (This part has not been covered in class), III.5, III.8 through III.8.1, IV.1, III.7.6, III.12.1-2. (All the exercises in this part of the assignment are for reference only; you may use them and do not have to solve them, unless you are asked to below or want to.)

Problem 1. Let $\mathcal{C}$ be the sheaf of continuous real-valued functions on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with the topology induced from $\mathbb{C}$. Compute the Čech cohomology groups $\check{H}^q(S^1, \mathcal{C})$, $q \geq 0$. *Hint*: Compute the Čech cohomology groups $H^q(\mathcal{U}_n, \mathcal{C})$, where $\mathcal{U}_n$, $n > 0$, consists of the open sets

$$\{z = e^{\pi i t/n} \mid m < t < m + 2\} \subset S^1, \qquad m = 0, \dots, 2n - 1.$$

Show that the coverings of this type are *coinitial* in the set of all open covers of S^1, *i.e.*, for every open cover $\mathcal{U}$, there exists an n such that $\mathcal{U}_n$ refines $\mathcal{U}$.

Problem 2. Hartshorne's Exercise III.4.11.

Problem 3. Compute $H^q(X, \mathcal{O}_X)$, $q \geq 0$, for the affine line $X = \overline{\mathbb{A}}^1_k$ over a field k with a doubled origin, for example, by providing justification for the previous homework's computation of Čech cohomology to produce Grothendieck cohomology $H^q(X, \mathcal{O}_X)$.

Problem 4. Compute $H^1(X, \mathcal{O}_X)$ for the affine plane $X = \mathbb{A}^2_k \setminus \{0\}$ over a field with the deleted origin 0 and conclude that X cannot be affine.

Problem 5. Use the long exact sequence for *Grothendieck cohomology* (*i.e.*, cohomology as a derived functor) and its relation to Čech cohomology (Cartan's theorem) to deduce the following statement, similar to Vakil's Exercise 18.2.C. Given a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

of $\mathcal{O}_X$-modules on a separated scheme X with $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ quasi-coherent, show that there is a long exact sequence for Čech cohomology $\check{H}^\bullet(X, -)$. Challenge: find an argument to see that only the quasi-coherence of $\mathcal{F}$ is enough to conclude the result.

Problem 6. Hartshorne's Exercise III.5.5 (a,c,d).

Problem 7. Vakil's Exercise 12.2.B (You may assume Theorem 11.3.3 and Exercise 12.1.B. There the *tangent space* at a point p of a scheme X is the linear dual k-space $(\mathfrak{m}/\mathfrak{m}^2)^*$ of the *cotangent space* $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$ and $k = \mathcal{O}_{X,p}/\mathfrak{m}$ is the residue field of p.)

Problem 8. Vakil's Exercise 18.4.D. (A *rational function* on an integral scheme X is an element of the fraction field $K(A)$ of A for an open affine subset $U = \operatorname{Spec} A$ of X. A *rational section* of an invertible sheaf $\mathcal{L}$ is an element in $\Gamma(U, \mathcal{L}) \otimes_A K(A)$. You may assume that your projective curve is integral, as well as the previous exercises in Section 18.4.)