MATH 8254: ALGEBRAIC GEOMETRY
PROBLEM SET 3, DUE MONDAY, MARCH 24, 2008

SASHA VORONOV

Problem 1 (Diagram Chase practice)**.** Complete the diagram chase I briefly indicated in class to construct a homomorphism between the $H^1(B^\bullet, d)$ and $H^1(A^\bullet, \delta)$, where $B^\bullet = \Gamma(X, C^\bullet(\mathcal{F}))$ and $A^\bullet = \Gamma(X, \mathcal{G}^\bullet)$, $(C^\bullet(\mathcal{F}), d)$ being the canonical flabby resolution of $\mathcal{F}$ and $(\mathcal{G}^\bullet, \delta)$ being a given flabby resolution of $\mathcal{F}$. That is to say, construct a well-defined map $f : \mathrm{Ker} d^1/\mathrm{Im} d^0 \to \mathrm{Ker}\delta^1/\mathrm{Im}\delta^0$, track all the ambiguities you make using the diagram choice to see that they will all result in a unique map to the quotient group $\mathrm{Ker}\delta^1/\mathrm{Im}\delta^0$, use this to see that you get a unique group homomorphism and that the homomorphism $g : \mathrm{Ker}\delta^1/\mathrm{Im}\delta^0 \to \mathrm{Ker} d^1/\mathrm{Im} d^0$ which you construct similarly is an inverse of f.

Problem 2 (Compare to H: Lemma III.2.10)**.** Let Y be a closed subspace of a topological space X and $i : Y \to X$ be the natural inclusion map. For a flabby sheaf $\mathcal{F}$ on Y, prove that $i_*\mathcal{F}$ is a flabby sheaf on X.

Problem 3 (Compare to H: Theorem III.1.1A (c))**.** Complete the proof of the Long Exact Sequence Theorem in class: If

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is an exact sequence of sheaves, then there is a long exact sequence of cohomology, as follows:

$$\begin{aligned} 0 \quad & \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \to \\ & \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to H^1(X,\mathcal{H}) \to \\ & \to H^2(X,\mathcal{F}) \to H^2(X,\mathcal{G}) \to H^2(X,\mathcal{H}) \to \quad \dots \end{aligned}$$

Problem 4. For sheaves $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{O}_X$-modules on a separated scheme X, define a pairing

$$H^p(X,\mathcal{F}) \otimes H^q(X,\mathcal{G}) \to H^{p+q}(X, \mathcal{F} \otimes \mathcal{G})$$

using Čech cohomology.

Problem 5 (Compare to H: Exercise III.4.5)**.** For a scheme X, let $\mathrm{Pic}X$ be the group of isomorphism classes of invertible shaves, called the *Picard group.* Show that $\mathrm{Pic}X \cong \check{H}^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units of the ring $\Gamma(U, \mathcal{O}_X)$, with multiplication as the group operation. Note that the group structure on the cohomology group $\check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$ induced by an open covering $\mathcal{U} = \{U_i \mid i \in I\}$ of X is defined by $\{f_{ij}\} \cdot \{g_{ij}\} := \{f_{ij} \cdot g_{ij}\}$ for cochains $\{f_{ij}\}$ and $\{g_{ij}\}$. The group structure on $\check{H}^1(X, \mathcal{O}_X^*)$ is then induced by inductive limit.

Date: March 9, 2008.

Problem 6. Hartshorne: III.4.3.

Problem 7. Hartshorne: III.4.7.