MATH 8254: ALGEBRAIC GEOMETRY
PROBLEM SET 4, DUE WEDNESDAY, APRIL 23, 2008

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Problem 1. Using the same method as we used in class, prove directly, without reduction to $m = 0$, that for any integer m, $H^p(X, \mathcal{O}_X(m)) = 0$ for $0 < p < n$, where $X = \mathbb{P}^n_R$ is the projective space over a Noetherian ring R. [The method we used in class was presenting an arbitrary Čech p-cocycle

$$\left(\frac{f_{i_0 i_1 \dots i_p}}{(x_{i_0} x_{i_1} \dots x_{i_p})^r}\right), \quad i_0 < i_1 < \dots < i_p, \quad f_{i_0 i_1 \dots i_p} \in S_{(r(p+1))},$$

in $C^p(\mathcal{U}, \mathcal{O}_X)$ explicitly as a Čech coboundary. If you do not have notes to remind you how it works, ask your classmates for notes or talk to me.]

Problem 2. For a closed immersion $f : X \to Y$, show that $R^p f_* \mathcal{F} = 0$ for $p > 0$, where $\mathcal{F}$ is a quasi-coherent sheaf on X. [Hint: For an affine open $U \subset Y$, note that $H^p(f^{-1}(U), \mathcal{F}) = 0$ for $p > 0$.]

Problem 3. Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be open coverings of a scheme X. Find a necessary and sufficient condition for $\{(f_i, U_i)\}$, $f_i \in \Gamma(U_i, \mathcal{K}^*)$, and $\{(g_j, V_j)\}$, $g_j \in \Gamma(V_j, \mathcal{K}^*)$, to determine the same element in $H^0(X, \mathcal{K}^*/\mathcal{O}^*)$. [Hint: Give an answer in terms of a refinement of the two coverings.]

Problem 4. When a Cartier divisor $\mathcal{D} = \{(f_i, U_i) \mid i \in I\}$ over an integral scheme X can be chosen as $f_i \in \Gamma(U_i, \mathcal{O})$, $\mathcal{D}$ is said to be an *effective* Cartier divisor, and we write $\mathcal{D} \geq 0$. In Proposition III.6.11, show that there is a bijection between effective Weil divisors and effective Cartier divisors.

Problem 5. For a separated Noetherian integral scheme X regular in codimension one and a Weil divisor D on it, let

$$\mathbb{L}(D) := \{f \in k(X) \mid f = 0 \text{ or } (f) + D \geq 0\}.$$

Show that $\mathbb{L}(D)$ is an additive group and $\mathbb{L}(D) \cong \mathbb{L}(E)$ for $D \sim E$.

Date: April 14, 2008.