MATH 8254: ALGEBRAIC GEOMETRY II
HOMEWORK 5
DUE SUNDAY, APRIL 9, 11:59 P.M.

INSTRUCTOR: SASHA VORONOV

Review your course notes, study *Vakil's* Sections 15.2.4, 15.2.8, 15.2.K, 15.2.10, 15.2.13, 15.2.H, 17.4 through 17.4.6, 17.6 through 17.6.3, and *Gathmann's* '2002/03 Section 9.3. Perhaps, reading my clarification text `http://www-users.cse.umn.edu/~voronov/8254s23/pdfs/ratl.pdf` before moving on to reading Section 15 of Vakil and watching the recording of the 3/31 lecture on Canvas should help. Do **Problem A** below and **Exercises from [Vakil]**: 17.4.A, 17.4.B, 17.6.B, 17.6.E. **Here I mean Vakil's text version of 12/31/2022.**

Problem A: Let A be an abelian group and X a topological space. The *constant sheaf* $\underline{A}$, defined by $\underline{A}(U) := C^0(U, A)$, where $C^0(U, A)$ is the additive group of continuous functions $U \to A$, where A is considered with discrete topology. ($C^0(U, A)$ is the same as the group of locally constant functions.) Another definition is $\underline{A}$ is the sheafification of the *constant presheaf*, which just assigns A to each U.

A sheaf $\mathcal{F}$ on X is *flabby*, a.k.a. *flasque*, if for each open $U \subset X$, $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. The following sequence of exercises implies that $H^p(X, \mathcal{K}_X^*) = \{0\}$ for $p > 0$ on an integral Noetherian scheme. Because of the long exact sequence of cohomology coming from

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0,$$

this implies that $\operatorname{CaCl} X \cong \operatorname{Pic} X$. (This isomorphism actually holds more generally for reduced Noetherian schemes.) There is a bit of cheating in the process, as in general, (Čech) cohomology of sheaves depends on the choice of an open cover and is independent of the cover, when we deal with quasi-coherent sheaves on (quasi-)compact, separated schemes and use affine open covers. However, when the pth cohomology group vanishes for every open cover, it will obviously be independent of the choice of an open cover. ☺

(1) Let X be an integral Noetherian scheme. Show that the *sheaf of rational functions* $\mathcal{K}_X$ (the sheafification of $\mathcal{K}'_X$, where $\mathcal{K}'_X(U)$ is the localization of the ring of regular functions $\mathcal{O}_X(U)$ on an open subset of U with respect to functions which are non-zero-divisors in each stalks on U) on a scheme X is constant (that is, $\underline{A}$ for $A = K(X) = \mathcal{K}_X(X)$, where $K(X)$ is the field of rational functions on X). Deduce also that the sheaf of invertible rational functions $\mathcal{K}_X^*$ is constant.
(2) Show that on an irreducible topological space X every constant sheaf is flasque.
(3) Show, directly using Čech cohomology (see, e.g., the construction of cohomology in Gathmann's Chapter 16, which we used for varieties and schemes, but defines Čech cohomology for any open cover of any space) adopted by us, that $H^p(X, \mathcal{F}) = 0$ for all $p > 0$ for any flasque sheaf $\mathcal{F}$ on any open cover of any topological space.

Date: April 1, 2023; Updated with Vakil's version clarification.

Submit the homework by the end of the due day, 11:59 p.m. Please submit it electronically to Gradescope at
`https://www.gradescope.com/courses/500650`,
which you can access through Canvas.