Math 8271 **Homework 5**

Posted: 11/11/2015; Updated: 11/17/2015; Due: Friday, 11/20/2015

The problem set is due at the beginning of the class on Friday, next week.

Reading: Sections 4.7 and 5.1.

Conventions: For this homework, G is a compact real Lie group, dg is the Haar measure on G, and V is a finite-dimensional complex representation of G.

Problem 1. Problem 4.4 (1,3) from the text.

Problem 2. Prove that for a function $f : G \to \mathbb{C}$, the functions $(R_g f)(h) := f(hg)$, $g \in G$, span a finite-dimensional complex vector space, iff f is a linear combination of (generalized) matrix coefficients of a representation.

Problem 3. Show that if V is an irreducible representation and χ its character, then

$$\int_G \chi(g) dg = \begin{cases} 1 & \text{if } V \cong \mathbb{C} \text{ is the trivial representation,} \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4. Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation and χ the character of V, Let V^G be the subspace of G-*invariants*, *i.e.*,

$$V^G := \{v \in V \mid \rho(g)v = v \text{ for all } g \in G\}.$$

Show that

$$\int_G \chi(g) dg = \dim(V^G).$$

Problem 5. Let $\rho_1 : G \to \mathrm{GL}(V_1)$, $\rho_2 : G \to \mathrm{GL}(V_2)$ be representations of G. Show that the G-bimodules $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ and $V_1^* \otimes V_2$ are isomorphic.

Problem 6. Let $\rho_1 : G \to \mathrm{GL}(V_1)$, $\rho_2 : G \to \mathrm{GL}(V_2)$ be representations of G and χ_1, χ_2 their characters. Show that

$$\int_G \chi_1(g)\overline{\chi_2(g)} dg = \dim \mathrm{Hom}_G(V_1, V_2).$$

Problem 7. Show that G has a finite-dimensional representation ρ which is *faithful*, *i.e.*, $\mathrm{Ker}\, \rho = 1$. [*Hint*: Apply the Peter-Weyl theorem to a function which is 0 at 1 and greater than 1 outside of a neighborhood of 1. Where will the kernel of a representation whose matrix coefficient (or a linear combination of such) approximating this function lie?]

Problem 8. Let $E_{ij} \in \mathfrak{gl}(n, \mathbb{R})$ be the elementary matrix, whose only nonzero entry is at the (i, j) position. Show that $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$. Using this, show that, for any natural number d,

$$\sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E_{i_1 i_2} E_{i_2 i_3} \ldots E_{i_d i_1}$$

is in the center of the universal enveloping algebra $U\mathfrak{gl}(n, \mathbb{R})$.