Posted: 1/23/2016; Updated 2/4/2016; Due: Friday, 2/5/2016
The problem set is due at the beginning of the class on Friday, February 5.

Reading: Class notes, consult with the Wikipedia articles on Maurer-Cartan forms and connections on principal G-bundles. Text: Review Sections 5.6, 5.8, and 6.1. Read Sections 6.2-3.

Problem 1. Calculate the Maurer-Cartan form on $G = S^1$.

Problem 2. For a Lie subgroup $H \subset G$, show that the restriction $\omega_G|_H$ of the Maurer-Cartan form ω_G on G to H is the Maurer-Cartan form ω_H on H:

$$\omega_G|_H = \omega_H.$$

Problem 3. For the Maurer-Cartan form $\omega = (x_{ij})^{-1}(dx_{ij})$ on $\mathfrak{gl}(n, \mathbb{R})$, prove the Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

by direct computation.

Problem 4. Consider the principal H-bundle $G \to G/H$ for a closed Lie subgroup H of a real Lie group G. Note that the Maurer-Cartan form does not readily produce a connection on that principal H-bundle, because ω is in $\Omega^1(G; \mathfrak{g})$, whereas a connection on the principal H-bundle is supposed to be in $\Omega^1(G; \mathfrak{h})$. Describe a construction which might allow you to construct a connection on this principal H-bundle using the Maurer-Cartan form ω on G. Check that the $\mathfrak{h}$-valued form that you construct is indeed a connection on the principal H-bundle. Find a condition under which this connection is flat, *i.e.*, has zero curvature.

Problem 5. Let G be a connected real Lie group with a Lie algebra whose Killing form is negative definite. Show that the adjoint representation is orthogonal, *i.e.*, admits a G-invariant inner product. **Note**: Do not use the fact that G must be compact, which we actually have never proven and are not going to. This problem spells out a claim made in the proof of Theorem 6.10.

Problem 6. Let G be a connected real Lie group with a semisimple Lie algebra. Show that

$$\operatorname{Ad} G = (\operatorname{Aut} \mathfrak{g})^0.$$

Note: This will complete the argument in Theorem 6.10 by implying that, provided that the Killing form is negative definite, $\operatorname{Ad} G$ is a closed subgroup in $\operatorname{SO}(\mathfrak{g})$ and thereby compact.

Problem 7. Problem 6.1 from the textbook (Section 6.8). **Note**: There is a sign error in the problem: under the conventions of the textbook, the Casimir C must be $-(J_x^2 + J_y^2 + J_z^2)/2$. Also, instead of comparing the conclusion to the proof of Lemma 4.62, compare it to the direct verification, as in Example 5.6, of the fact that $J_x^2 + J_y^2 + J_z^2$ is central, which we did in class last term. "Compare" means nothing for your actual homework: it is just a motivational remark for you. All you need to do is to show that $-(J_x^2 + J_y^2 + J_z^2)/2$ is the Casimir.

Problem 8. Why does the trivial extension $0 \to V_2 \to V_1 \oplus V_2 \to V_1 \to 0$ represent the zero element in the Abelian group $\operatorname{Ext}^1_{\mathfrak{g}}(V_1, V_2)$ of extensions $0 \to V_2 \to W \to V_1 \to 0$ of representations of a Lie algebra $\mathfrak{g}$? The group law on $\operatorname{Ext}^1_{\mathfrak{g}}$ is given by the Baer sum.