Math 8272 **Homework 3**
Posted: 2/22/2016; Updated: 3/3/2016; Due: Friday, 3/4/2016

The problem set is due at the beginning of the class on Friday, March 4.

Reading: Text: Sections 6.6 after Example 6.40, 7.1-4, 7.6 through Definition 7.22.

Convention: Unless stated to the contrary, root systems are assumed to be *reduced* in the text and on the homework. In particular, as Zeshen noticed, Problem 7.2(2) would not be true for the non-reduced system of Problem 7.1. However, the statement of 7.2(2) is correct, if the root system is assumed to be reduced.

Problem 1. Show that every element in the Lie algebra $\mathfrak{g}$ of a *compact* Lie group is semisimple, *i.e.*, for each $x \in \mathfrak{g}$, the operator $\operatorname{ad} x : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is diagonalizable. (This means that if we were looking at maximal toral subalgebras in $\mathfrak{g}$, we could drop the condition of semisimplicity.)

Problem 2. Show that if $R \subset E$ is a root system, then

$$R' := \{\alpha/(\alpha, \alpha) \mid \alpha \in R\}$$

is also a root system.

Problem 3. Show that for a semisimple lie algebra $\mathfrak{g}$, the elements $H_\alpha \in \mathfrak{h}$ dual to the roots $\alpha \in R$:

$$(H_\alpha, h) = \alpha(h), \qquad h \in \mathfrak{h},$$

form a root system. [*Hint*: Do Problem 7.2 first.]

Problem 4. Suppose that S is any subset of a polarized root system $R = R_+ \coprod R_-$ such that if $\alpha \in R$, then either $\alpha \in S$ or $-\alpha \in S$. Assume also that if $\alpha, \beta \in S$ and $\alpha + \beta \in R$ then $\alpha + \beta \in S$. Show that there exists $w \in W$ such that $w(S) \supset R_+$. Show also that if S is such that for any $\alpha \in R$ either $\alpha \in S$ or $-\alpha \in S$ but never both, then w is unique. You may assume that $w(R_+) \subset R_+$ implies $w = 1$, which we will prove later. [Given how much we know, this is a hard problem. Here is a *hint*: First, prove that for a reflection s_α in a simple root α, $s_\alpha(R_+) = R_+ \setminus \{\alpha\} \cup \{-\alpha\}$. Even though this statement appears further on in the text as a consequence of more advanced stuff, it is easy to prove this by methods we have learned so far. Then use a suitable reflection s_α to increase the number of positive roots in S, if need be, by replacing S with $s_\alpha(S)$.]

Problems from Chapter 7 (Section 7.11): **1, 2** (Replace (7.17) with (7.7) in the hint), **3** (Do not forget to use the corrected version of Lemma 7.17, as per the online *Errata*), and **16**.