Math 8272 **Homework 5**
Posted: 4/6/2016; Updated: 4/11; Due: Friday, 4/15

The problem set is due at the beginning of the class on Friday, April 15, 2016.

Reading: Text: Sections 7.9, 8.1–2.

Problem from Chapter 7 (Section 7.11): **13**.

Problem from Chapter 8 (Section 8.10): **1, 3** [*Hint*: Solve Problem 1 below first], **9**.

Problem 1. Let P be the weight lattice of a polarized, reduced root system R in a Euclidean space E with a set $\{\alpha_1, \dots, \alpha_r\}$ of simple roots. Let $\omega_i \in E$ be the *fundamental weights*, defined by

$$\omega_i(\alpha_j^\vee) = \delta_{ij}, \qquad i, j = 1, \dots, r,$$

where the (simple) *coroots* $\alpha_i^\vee \in E^*$ are defined by

$$\lambda(\alpha_i^\vee) = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \qquad \text{for all } \lambda \in E.$$

Show that the weight lattice P is the free Abelian group on the fundamental weights.

Problem 2. (1) Take the B_n root system R in the Euclidean space $\mathbb{R}^n$ with a standard basis $\{e_1, \dots, e_n\}$, see page 205 (Appendix A.2). Show that its Dynkin diagram is indeed what it is supposed to be.
(2) Do the same for the D_n root system from page 208, Appendix A.4. (Note that there is a misprint in the Dynkin diagram on page 208, see the *Errata* or page 153.)

Problem 3. Suppose that we have a connected Dynkin diagram corresponding to a reduced root system. Show that no more than three edges can be incident to a given vertex. Deduce that the only reduced root system with a connected Dynkin diagram and at least one triple edge is G_2. [You are not supposed to use the classification of Dynkin diagrams here. This is rather one of the steps in proving that theorem in the non simply-laced case.]

Problem 4. Fix a finite-dimensional complex semisimple Lie algebra. Prove a universality property of the Verma module M_λ of highest weight λ with a highest weight vector v_λ: given a highest weight representation V of highest weight λ with a highest-weight vector v, there is a unique morphism of representations $M_\lambda \to V$ mapping v_λ to v. Show that this morphism will automatically be onto.