MATH 8306: ALGEBRAIC TOPOLOGY DUALITY BETWEEN HOMOLOGY AND COHOMOLOGY WITH FIELD COEFFICIENTS

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Here is brushed-up proof of duality between homology and cohomology with field coefficients.

Theorem 1. *If k is a field and K a simplicial complex, then*

$$H^n(K;k) = (H_n(K;k))^*$$

for all values of $n \geq 0$.

Proof. Step 1. For each $n \geq 0$, we have a natural isomorphism $C^n(K;k) = C_n(K;k)^*$ or $\mathrm{Hom}_{\mathbb{Z}}(C_n(K;\mathbb{Z}),k) = \mathrm{Hom}_k(C_n(K;k),k)$, because $C_n(K;\mathbb{Z})$ is a free abelian group. It is freely generated by the set K_n of n-simplices of K.

Step 2. Now we see that the cochain complex

$$\dots \to C^{n-1}(K;k) \xrightarrow{\delta^n} C^n(K;k) \xrightarrow{\delta^{n+1}} C^{n+1}(K;k) \to \dots$$

is a complex of vector spaces linear dual to the chain complex

$$\dots \to C_{n+1}(K;k) \xrightarrow{\partial_{n+1}} C_n(K;k) \xrightarrow{\partial_n} C_{n-1}(K;k) \to \dots$$

What remains to be shown is that the vector space $H^n(K;k) = \operatorname{Ker}\delta^{n+1}/\operatorname{Im}\delta^n$ is the linear dual of $H_n(K;k) = \operatorname{Ker}\partial_n/\operatorname{Im}\partial_{n+1}$.

Step 3. A remarkable thing about complexes of vector spaces is that given a short exact sequence (SES) $0 \to U \to V \to W \to 0$ of vector spaces, the linear dual sequence $0 \to W^* \to V^* \to U^* \to 0$ is also exact. This can be easily shown by *splitting* the given SES, that is to say, presenting V as the direct sum of U and a complementary subspace isomorphic to W, which gives $V \cong U \oplus W$ and thereby $V^* \cong U^* \oplus W^*$, yielding $0 \to W^* \to V^* \to U^* \to 0$. Such a complementary subspace always exists: complete a basis of U to a basis of V, and take the linear span of the basis vectors which are not in U. This works even for infinite dimensional vector spaces, just requires the axiom of choice. Our simplicial complexes need not be finite: K_n, which forms a basis of $C_n(K;k)$, could well be an infinite set of simplices (*e.g.*, triangulate $\mathbb{R}^2$ by tiling it into triangles).

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Step 3 is where the argument would break, should we try to use it for abelian groups: dualize $0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \to \mathbb{Z}_2 \to 0$, that is to say, apply $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ or $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}_2)$ and see what happens. This is why there is no duality like that for homology and cohomology with arbitrary abelian coefficients.

Step 4. We have an SES

$$0 \to \operatorname{Im} \partial_{n+1} \to \operatorname{Ker} \partial_n \to H_n \to 0$$

This dualizes to an SES

$$0 \to H_n^* \to (\operatorname{Ker} \partial_n)^* \to (\operatorname{Im} \partial_{n+1})^* \to 0.$$

Step 5: $(\operatorname{Im} \partial_{n+1})^* = \operatorname{Im} \delta^{n+1}$, because the linear map $C_{n+1} \xrightarrow{\partial_{n+1}} C_n$ can be factored into $\partial_{n+1} : C_{n+1} \xrightarrow{\partial_{n+1}} \operatorname{Im} \partial_{n+1} \hookrightarrow C_n$, which dualizes to $\delta^{n+1} : C^n \twoheadrightarrow (\operatorname{Im} \partial_{n+1})^* \xrightarrow{\delta^{n+1}} C^{n+1}$.

Step 6. $(\operatorname{Ker} \partial_n)^* = C^n / \operatorname{Im} \delta^n$, because the SES

$$0 \to \operatorname{Ker} \partial_n \to C_n \to \operatorname{Im} \partial_n \to 0$$

dualizes to an SES

$$0 \to (\operatorname{Im} \partial_n)^* \to C^n \to (\operatorname{Ker} \partial_n)^* \to 0$$

and $(\operatorname{Im} \partial_n)^* = \operatorname{Im} \delta^n$, as we have seen earlier (for any n).

Step 7. Collecting the last two steps, we get an SES

$$0 \to H_n^* \to C^n / \operatorname{Im} \delta^n \to \operatorname{Im} \delta^{n+1} \to 0$$

with the last linear map being induced by δ^{n+1}. Thus, H_n^* may be naturally identified with the kernel of this map, which is obviously $\operatorname{Ker} \delta^{n+1} / \operatorname{Im} \delta^n = H^n$. □