Math 8306 **Homework 4**

Posted: 10/21; Updated 10/27; Due: Friday, 10/28/2016

The problem set is due at the beginning of the class on Friday, October 28.

Reading: Class notes. Text: Sections 2.2 (138–141, 144, 146–147, 153), 2.C (179–181), and 3.3 (239–242, there done differently, via cap product)); May's Concise Course: Sections 21.1 and 21.3.

Conventions: No coefficients means integral coefficients: $H_n(K) := H_n(K; \mathbb{Z})$. A *closed manifold* is a smooth, compact manifold with no boundary.

Problem 1. Let M be closed, orientable, four-dimensional manifold with $H_1(M)$ free as an abelian group. Show that all the homology groups are free.

Problem 2. Let M be a closed, oriented n-dimensional manifold and $\lambda : H_k(M) \otimes H_{n-k}(M) \to \mathbb{Z}$ be given by the intersection number:

$$\lambda(\sum_i a_i \sigma_i, \sum_i b_i \sigma_i^*) = \sum_i a_i b_i,$$

where σ_i runs over the k-simplices of a nice triangulation of M and σ_i^* is the dual cell in the dual CW complex.

(1) Show that the natural homomorphism $\alpha : H_{n-k}(M) \to \mathrm{Hom}(H_k(M); \mathbb{Z})$ induced by λ:

$$\alpha : c \mapsto \lambda(-, c)$$

is surjective with torsion kernel. [*Hint*: Use the Universal Coefficient theorem.]

(2) Show that when the integral coefficients are replaced with those in a field α becomes an isomorphism.

Problem 3. For K and L finite simplicial complexes and $p : K \to L$ an n-sheeted covering space, show that $\chi(K) = n\chi(L)$.

Problem 4. For finite simplicial complexes K and L, show that $\chi(K \times L) = \chi(K)\chi(L)$. [*Hint*: It may be wise to use cellular homology with field coefficients here. You may assume it is produced by tensoring the integral cellular chain complex with the group of coefficients, so that cellular homology with coefficients is isomorphic to simplicial homology with coefficients.]

Problem 5. Suppose K is a finite simplicial complex and $f : K \to K$ is a continuous map such that its nth iteration f^n is the identity. Prove that if $\chi(K) \not\equiv 0 \pmod{n}$, then at least one of $f, f^2, \dots, f^{n-1}$ has a fixed point.

Problem 6. Prove that the Euler characteristic of the complement to a tame knot in S^3 is zero.

Problem 7. If K is a finite simplicial complex and $f : K \to K$ is a simplicial homeomorphism, show that the Lefschetz number Λ_f equals the Euler characteristic of the set of fixed points of f. In particular, Λ_f is the number of fixed points if the fixed points are isolated. [*Hint*: Barycentrically subdivide K to make the fixed point set a subcomplex.]