Math 8306 **Homework 6**

Posted: 11/26; Updated 12/08; Due: Friday, 12/09/2016

The problem set is due at the beginning of the class on Friday, December 9.

Reading: Class notes. Text: Sections 2.3 (160–165), 3.1 (197–202), 3.2 (Example 3.7), 2.1 (128–130) and 3.B (268–273, done using cellular homology), Matthew Ando's lecture on Alexander-Whitney online (Lecture Notes 5), Section 3.B (273–275).

Conventions: Homology by default means *singular homology.* No coefficients means integral coefficients: $H_n(X) := H_n(X;\mathbb{Z})$.

Problem 1. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not. *Hint*: The universal covering space of $S^1 \vee S^1 \vee S^2$ is the tree on p. 59 (Section 1.3) of Hatcher with a copy of S^2 attached at each vertex.

Problem 2. Compute the homology $H_k(T^n)$ of the torus $T^n = (S^1)^n$.

Problem 3. Use the Künneth theorem to show that the product of two closed manifolds is orientable, if and only if both are.

Problem 4. Prove that the sphere S^n, $n \geq 0$, is not the product of two manifolds of positive dimension.

Problem 5. Compute the homology of $S^m \times \mathbb{R}P^n$. You may assume the homology of each factor to be known.

Problem 6. Repeat the proof of the algebraic Künneth theorem to show that for two nonnegative chain complexes $C_\bullet$ and $C'_\bullet$ of vector spaces over a field k, $H_n(C_\bullet \otimes_k C'_\bullet) = \bigoplus_{i+j=n} H_i(C_\bullet) \otimes_k H_j(C'_\bullet)$. What can you say about $H_n(X \times Y; k)$ for two topological spaces X and Y?

Problem 7. Show that the cup product

$$H^2(S^2 \vee S^4) \otimes H^2(S^2 \vee S^4) \to H^4(S^2 \vee S^4)$$

is zero. *Hint*: Consider a map $f : S^2 \vee S^4 \to S^2$.

Problem 8. Let $\Delta[k]$ be the simplicial set with

$$\Delta[k]_n := \{(i_0, i_1, \dots, i_n) \mid i_j \in \mathbb{Z} \text{ and } 0 \leq i_0 \leq i_1 \leq \cdots \leq i_n \leq k\},$$

where d_j and s_j are defined by dropping and repeating the jth component. Show that the geometric realization $|\Delta[k]|$ of $\Delta[k]$ is homeomorphic to the standard k-simplex Δ^k. Just find a way to identify them, *i.e.*, show there is a bijection, do not worry about proving it is a homeomorphism, as long as you convince yourself that it is. *Hint*: For any simplicial set $X_\bullet$, its geometric realization $|X_\bullet| = \coprod_{n\geq 0} X_n \times \Delta^n / \{(s_i x, u) \sim (x, \sigma_i u), (d_i x, u) \sim (x, \delta_i u)\} = \coprod_{n \geq 0} X_n^{\text{nondeg}} \times \Delta^n / \{(d_i x, u) \sim (x, \delta_i u)\}$, where $X_n^{\text{nondeg}} := X_n \setminus \bigcup_{i=0}^{n-1} s_i(X_{n-1})$ is the set of "non-degenerate simplices" in X_n, obtained by removing the points in X_n which may be represented as $s_i x$ for some i and x. This identification of $|X_\bullet|$ comes from taking the quotient by the relation $(s_i x, u) \sim (x, \sigma_i u)$ first and noticing that every degenerate simplex is a denercy (the result of applying a sequence of degeneracy maps) of a unique nondegenerate simplex via a unique degeneracy. This means that in the geometric realization, a degenerate simplex gets collapsed onto a unique smaller-dimensional simplex by a unique map.