Appendix (29.23A) Acyclic Models

In this appendix we treat the Eilenberg-Mac Lane method of acyclic models (see [22], or Mac Lane [38], Chapter VIII) and apply it to the theory of products.

We have seen that to an ordered pair (X, Y) of topological spaces we can associate

$$F(X, Y) = S(X \times Y),$$

$$F'(X, Y) = S(X) \otimes S(Y)$$

which are *augmented chain complexes*: An *augmentation* of a chain complex C of R is an epimorphism $\varepsilon: C_0 \to R$ such that the composite

$$C_1 \xrightarrow{\partial} C_0 \xrightarrow{\varepsilon} R$$

is zero; the augmentation of $S(X \times Y)$ is $\partial^{\#}$ (9.7), while that of $S(X) \otimes S(Y)$ is

$$\partial^{\#} \otimes \partial^{\#}: S_0(X) \otimes S_0(Y) \to R \otimes R \xrightarrow{\simeq} R$$

We can make a category $\mathscr{P}$ out of the ordered pairs (X, Y) by defining a morphism $(X, Y) \to (X', Y')$ to be a map of the form $f \times g$, where $f: X \to X'$, $g: Y \to Y'$. We can make a category $\mathscr{C}$ out of the augmented chain complexes by defining a morphism $h: C \to C'$ to be a chain homomorphism which preserves the augmentations, i.e., $\varepsilon' h = \varepsilon$. Then F is clearly a functor from $\mathscr{P}$ to $\mathscr{C}$, and F' is also $(F'(f \times g) = S(f) \otimes S(g): S(X) \otimes S(Y) \to S(X') \otimes S(Y'))$.

Moreover, Alexander-Whitney gives us a morphism (natural transformation) of functors

$$A: F \to F'$$

as follows from (29.1), (29.3), and the fact that

$$A: S_0(X \times Y) \to S_0(X) \otimes S_0(Y)$$

preserves the augmentations (exercise). The Eilenberg-Zilber theorem can be rephrased to state that A is a *chain equivalence* of functors: This means that there exists a morphism of functors

$$B: F' \to F$$

such that AB and BA are chain homotopic to the identity morphisms. In general, if $\mathscr{A}$ is any category, $F, F': \mathscr{A} \to \mathscr{C}$ functors ($\mathscr{C}$ as above), $\Phi, \Psi: F \to F'$ morphisms of functors, we say Φ and Ψ are *chain homotopic* if for every object X in $\mathscr{A}$, the chain homomorphisms $\Phi(X), \Psi(X): F(X) \to F'(X)$ are chain homotopic by means of a chain homotopy which is functorial in X.

The technique of acyclic models gives us a sufficient condition for the two functors F, F' with values in $\mathscr{C}$ to be chain equivalent. In fact, the condition is so strong it imples that *any* morphism of functors $F \to F'$ is a chain equivalence.

Given a category $\mathscr{A}$ with a distinguished set of objects $\mathscr{M}$. We say that the pair $(\mathscr{A}, \mathscr{M})$ is a *category with models*, the *models* being the objects in $\mathscr{M}$. Let $F: \mathscr{A} \to \mathscr{C}$ be any functor. For each $q \geq 0$ we have a functor F_q from $\mathscr{A}$ to the category of R-modules which assigns to any X in $\mathscr{A}$ the q^{th} component of the chain complex $F(X)$. We say F_q has a *basis* (relative to $\mathscr{M}$) if there is an indexed family $[d_j \in F_q(M_j)\}_{j \in J}$, where the M_j are some of the models, such that for every X in $\mathscr{A}$, $F_q(X)$ is the free R-module spanned by all the elements

$$\{F_q(u)(d_j)\}_{j \in J}, \qquad \text{all } u: M_j \to X$$

We say F is *free* if all the F_q's have bases.

(29.24) *Example*: Let $F, F': \mathscr{P} \to \mathscr{C}$ be the two functors of the Eilenberg-Zilber theorem. As models for $\mathscr{P}$ we take all the ordered pairs (Δ^p, Δ^q) (where Δ^p is the standard geometric p-simplex). Then F and F' are both free: *Let* $d_q: \Delta^q \to \Delta^q \times \Delta^q$ be the diagonal map, so that $d_q \in S_q(\Delta^q \times \Delta^q)$. For any ordered pair (X, Y), we know that the singular q-simplexes σ form a basis for the free R-modules $S_q(X \times Y)$. But the formula

$$\sigma = [(p_X \sigma) \times (p_Y \sigma)] \circ d_q$$

(where $p_X: X \times Y \to X$, $p_Y: X \times Y \to Y$ are the projections) shows that the singleton $\{d_q\}$ is a basis for F_q. Thus F is free relative to the chosen models. On the other hand, a basis for $F'_n(X, Y)$ consists of all $\sigma \otimes \tau$, σ a singular p-simplex in X, τ a singular q-simplex in Y, $p + q = n$. But the formula (see 9.10)

$$\sigma \otimes \tau = F'_n(\sigma \times \tau)(\delta_p \otimes \delta_q)$$

(where $\delta_p \in S_p(\Delta^p)$, $\delta_q \in S_q(\Delta^q)$ are the identity maps) shows that the family $\{\delta_p \otimes \delta_q\}_{p+q=n}$ is a basis for F'_n. Thus F' is free.

If C is an augmented chain complex with augmentation $\varepsilon: C_0 \to R$, we can form the reduced chain complex C given by

$$\tilde{C}_q = \begin{cases} C_q & q > 0 \\ \text{Kernel } \varepsilon & q = 0 \end{cases}$$

The homology modules of $\tilde{C}$ are called the *reduced* (*augmented*) *homology modules* of C. A functor $F:\mathscr{A} \to \mathscr{C}$ will be called *acyclic* (relative to $\mathscr{M}$) if for every model $M \in \mathscr{M}$, all the augmented homology modules of the chain complex $F(M)$ are zero.

(29.24) *Example* (continued): In the Eilenberg-Zilber situation, $\tilde{H}_n^{\#}(\Delta^p \times \Delta^q) = 0$ for all n, p, and q, since $\Delta^p \times \Delta^q$ is contractible. Thus F is acyclic. We also have $\tilde{H}_n^{\#}[S(\Delta^p) \otimes S(\Delta^q)] = 0$ for all n, p, and q: Let R be the chain complex which has R in dimension 0 and 0 in all other dimensions, with zero boundary operator. Then the chain homomorphism $\partial^{\#}:S(\Delta^p) \to R$ is a chain equivalence since Δ^p is contractible. Hence

$$\partial^{\#} \otimes \partial^{\#}:S(\Delta^p) \otimes S(\Delta^q) \to R \otimes R \cong R$$

is a chain equivalence, from which the assertion follows. Thus F' is also acyclic.

We can now state our main result.

(29.25) *Theorem on Acyclic Models. Let $(\mathscr{A},\mathscr{M})$ be a category with models, $\mathscr{C}$ the category of augmented chain complexes over R. Let F, $F':\mathscr{A} \to \mathscr{C}$ be functors such that F is free and F' is acyclic. Then there exists a morphism of functors $\Phi:F \to F'$ unique up to chain homotopy.*

(29.26) *Corollary. If both F and F' are free and acyclic, then F and F' are chain equivalent, and in fact any morphism of functors $F \to F'$ is a chain equivalence.*

Proof of Corollary: By the theorem there are morphisms $\Phi:F \to F'$ and $\Psi:F' \to F$. Since by the theorem there are unique morphisms $F \to F$ and $F' \to F'$ up to chain homotopy, $\Phi\Psi$ and $\Psi\Phi$ must be chain homotopic to the respective identity morphisms. ■

Using (29.24) we see that the Eilenberg-Zilber theorem follows at once.

(29.27) *Exercise*. An explicit chain homomorphism

$$B:S(X) \otimes S(Y) \to S(X \times Y)$$

which is functorial in (X, Y) is *the shuffle homomorphism given by*

$$B((\sigma \otimes \tau) = \sum \pm (D_{j_q} \ldots D_{j_1}\sigma, D_{i_p} \ldots D_{i_1}\tau)$$

where σ is a singular p-simplex, τ a singular q-simplex, D_k is the k^{th} degeneracy operator (proof of (24.8), part 2), the sum is over all permutations $(i_1 \ldots i_p j_1 \ldots j_q)$ of $(0 \ldots p + q - 1)$ such that $i_1 < i_2 < \cdots < i_p$, $j_1 < j_2 < \cdots < j_q$, and the sign is the signature of the permutation.

Before proving (29.25) we give some other applications.

For any ordered pair (X, Y), define a homomorphism

$$T: S(X) \otimes S(Y) \to S(Y) \otimes S(X)$$

by

$$T(z \otimes w) = (-1)^{pq} w \otimes z$$

for $z \in S_p(X)$, $w \in S_q(Y)$. Then T is a chain homomorphism:

$$\partial T(z \otimes w) = (-1)^{pq}[\partial w \otimes z + (-1)^q w \otimes \partial z]$$

$$T\partial(z \otimes w) = T[\partial z \otimes w + (-1)^p z \otimes \partial w]$$

$$= (-1)^{pq\ q} w \otimes \partial z + (-1)^{pq} \partial w \otimes z$$

Clearly T preserves augmentation and is functorial in (X, Y).

Consider the diagram

$$\begin{array}{ccc} S(X \times Y) & \xrightarrow{S(i)} & S(Y \times X) \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ S(X) \otimes S(Y) & \xrightarrow[T]{} & S(Y) \otimes S(X) \end{array}$$

where i is the interchange homeomorphism $X \times Y \to Y \times X$.

(29.28) *Proposition. This diagram is chain homotopy commutative, i.e., $AS(i)$ is chain homotopic to TA.*

The proof follows immediately from the uniqueness part of the acyclic model theorem. ■

(29.29) *Corollary. If $\zeta \in H_p(X)$, $\omega \in H_q(Y)$ then*

$$H_{p+q}(i)(\zeta \times \omega) = (-1)^{pq} \omega \times \zeta$$

(29.30) *Exercise.* Prove formulas (29.17.8) and (29.21) by the same method, i.e., deduce them from the chain homotopy commutativity of suitable diagrams, the latter being proved by referring to the theorem on acyclic models.

Proof of (29.25): Let $[d_j \in F_0(M_j)]_{j \in J_0}$ be a basis for F_0. Since F' is acyclic, the augmentation induces an isomorphism

$$\varepsilon' : H_0(F'(M_j)) \simeq R$$

for all $j \in J_0$. Hence there is a unique $z_j \in H_0(F'(M_j))$ such that

$$\varepsilon'(z_j) = \varepsilon(d_j)$$

for all $j \in J_0$.

There is a unique morphism of functors $\phi : H_0(F) \to H_0(F')$ preserving augmentation, namely the one sending the class of a linear combination

$$\sum \nu_{uj} F_0(u)(d_j) \in F_0(X)$$

($\nu_{uj} \in R$, $u : M_j \to X$) to the element

$$\sum \nu_{uj} H_0(F')(u)(z_j) \in H_0(F'(X))$$

We will construct a morphism $\Phi : F \to F'$ so as to give $\phi = H_0(\Phi)$. If Ψ is any morphism $F \to F'$, then by uniqueness $\phi = H_0(\Psi)$, and we will simultaneously construct the chain homotopy $D : \Phi \simeq \Psi$.

Thus for every object $X \in \mathscr{A}$ we must define a chain homomorphism $\Phi(X) : F(X) \to F'(X)$ (resp. a chain homotopy $D(X) : F(X) \to F'(X)$) such that for any $h : X \to Y$ we have

$$\Phi(Y)F(h) = F'(h)\Phi(X)$$

$$(\text{resp. } D(Y)F(h) = F'(h)D(X))$$

For $q \geq 0$, let $\{d_j \in F_q(M_j)\}_{j \in J_q}$ be a basis for F_q. Once we know the values $\Phi_q(M_j)(d_j)$ (resp. $D_q(M_j)(d_j)$) then for any X we are forced to define

$$\Phi_q(X)F_q(u)(d_j) = F'_q(u)\Phi_q(M_j)(d_j)$$

$$(\text{resp. } D_q(X)F_q(u)(d_j) = F'_q(u)D_q(M_j)(d_j))$$

which determines $\Phi_q(X)$ (resp. $D_q(X)$). These values are not arbitrary since we want

$$\partial \Phi_q(X) = \Phi_{q-1}(X)\partial$$

$$(\text{resp. } \partial D_q(X) = \Phi_q(X) - \Psi_q(X) - D_{q-1}(X)\partial)$$

We use induction on q. For $q = 0$, define $\Phi_0(M_j)(d_j)$ to be any representative in $F'(M_j)$ of $z_j \in H_0(F'(M_j))$ (resp. define $D_0(M_j)(d_j)$ to be any element of $F'_1(M_j)$ whose boundary is the element $\Phi_0(M_j)(d_j) - \Psi_0(M_j)(d_j)$ of kernel of the augmentation). Then for $w \in F_0(X)$ the homology class of $\Phi_0(X)(w)$ is obtained by applying $\phi(X)$ to the class of w. In particular, for $j \in J_1$, $\Phi_0(M_j)(\partial d_j)$ is a boundary in $F'_0(M_j)$, so define $\Phi_1(M_j)(d_j)$ to be any element such that

$$\partial \Phi_1(M_j)(d_j) = \Phi_0(M_j)(\partial d_j)$$

Assume $q > 1$ and Φ_p defined for $p < q$ so as to commute with ∂. Note that $\Phi_{q-1}(M_j)(d_j)$ is a cycle; since $H_{q-1}(F'(M_j)) = 0$, we can define $\Phi_q(M_j)(d_j)$ to be any element such that

$$\partial \Phi_q(M_j)(d_j) = \Phi_{q-1}(M_j)(d_j)$$

[Assume $q > 0$ and D_p defined for $p < q$ so as to be a chain homotopy. Note that for every $j \in J_q$, the element

$$\Phi_q(M_j)(d_j) - \Psi_q(M_j)(d_j) - D_{q-1}(M_j)(\partial d_j)$$

is a cycle. Since $H_q(F'(M_j)) = 0$, we can define $D_q(M_j)(d_j)$ to be any element whose boundary is this cycle.]. ∎

(29.31) *Remark.* We can view our results somewhat more categorically if we notice that $S(X)$ carries all the structure for the homology and cohomology of the space X. Thus far we have regarded $S(X)$ only as an augmented chain complex. To obtain the multiplicative properties, consider the augmentation preserving chain homomorphism

$$S(X) \xrightarrow{S(d)} S(X \times X) \xrightarrow{A} S(X) \otimes S(X)$$

where d is the diagonal map and A is Alexander-Whitney; let $m = AS(d)$. Then m is the *comultiplication* for a structure of *coalgebra* on $C = S(X)$, i.e., the diagram