Math 8306 **Fall 2021** **Homework 1**

Posted: 9/12; Problem 4 changed from the torus to $\mathbf{RP}^2$ on 9/15; due Monday, 9/20

The problem set is due at the beginning of the class on Monday (on paper or by email). A *ring* R will always mean a commutative ring with unit, unless specified otherwise.

Reading:
`http://www.math.umn.edu/~voronov/8306-21/syllabus.html`.
Class notes.
Hatcher: Chapter 2 (pages 97–103, skipping the notion of a Δ-complex, which we will not be using in the course, top of page 106, page 108, top paragraph of page 109, pages 111, 115–117, 5, 124, 137–141, 104–107, 128) and Chapter 3 (pages 218–219, 190–191, 197–201, simplicial and cellular cohomology on pages 202–203)]. Note the difference between the notion of a simplicial complex and that of a Δ-complex, discussed on p. 107.

Problem 1. Show that the boundary map $\partial : S_q(X, A; R) \to S_{q-1}(X, A; R)$, defining the relative singular chain complex, is well defined and squares to zero. Here $S_q(X, A; R) = S_q(X; R)/S_q(A; R)$ is the relative singular chain group of a pair (X, A) with coefficients in a ring R. You may assume ∂ is defined for the absolute singular chain complex $S_q(Y; R)$ for any space Y and that $\partial^2 = 0$ there.

Problem 2. Prove the cellular boundary formula describing the differential in cellular homology using the degree of a map between spheres by elaborating the argument that has been indicated in class on Friday, 9/10, (with a bit more coming on Monday, 9/13) or rephrasing the proof in the textbook on pp. 140–141.

Problem 3. Represent the torus T, pictured on pp. 5 and 102, as a cellular complex and compute its cellular homology with coefficients in a ring R.

Problem 4. Finish the computation of the simplicial homology of $\mathbf{RP}^2$ that we started in class on Wednesday, 9/15. This is similar to Example 2.4 on p. 106, but you will have to use a triangulation of the torus T (presentation of it as a simplicial complex), such as the one we used in Wednesday class.

Problem 5. Exercise 5 on p. 131 of Hatcher with the change that you will need to compute the simplicial homology of a triangulation of the Klein bottle, pictured on pp. 19 and 102, as in the previous problem. Do this for coefficients $R = \mathbb{Z}$, as well as $R = \mathbb{Z}/2$.

Problem 6. Calculate the tensor product of two finitely generated abelian groups.

Problem 7. Calculate the group $\operatorname{Hom}_{\mathbb{Z}}(A, B)$ of homomorphisms of two finitely generated abelian groups A and B.

Problem 8. Find a left adjoint to the forgetful functor R-**mod** $\to$ **Set**.

Problem 9. Identify $H^0(X; \mathbb{Z})$ in terms of the topology of X. Show that $H_0(\mathbb{Q}; \mathbb{Z}) \ncong H^0(\mathbb{Q}; \mathbb{Z})$, where the set $\mathbb{Q}$ of rationals is considered with the topology induced from the space $\mathbb{R}$ of reals. (You may observe that the connected components and, therefore, path components of $\mathbb{Q}$ are singletons, even though the topology is not discrete.)