**Math 8306** **Fall 2021** **Homework 2**
**Posted: 9/25; Corrected: 9/29; due Friday, 10/1**

The problem set is due at the beginning of the class on Friday (on paper or by email). A *ring* $R$ will always mean a commutative ring with unit, unless specified otherwise.

**Reading**:
**Class notes.**
**Hatcher**: Chapter 2 (pages 144, 160–165, free resolutions in the middle of p. 195 and Ext at the bottom of p. 195 and top of p. 196), Chapter 3 (pages 190–193, 199–202, 215), Chapter 3.A (page 265).

**Problem 1.** Prove the Hurewicz theorem for the fundamental group by elementary means, *i.e.*, construct a group homomorphism from the abelianization of the fundamental group of a path connected space to the first integral homology group along with an inverse homomorphism.

**Problem 2.** Compute the homology and cohomology of $\mathbf{RP}^2$ with coefficients in $\mathbb{Z}/2$ without using the Universal Coefficient Theorem, even if we cover it by then. Use the cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with coefficients in $\mathbb{Z}$ and treat $\mathbb{Z}/2$ as a $\mathbb{Z}$-module. (Note that homology and cohomology are isomorphic. We will later learn that this is always the case for field coefficients.)

**Problem 3.** For a short exact sequence of $R$-modules

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0,$$

show that the splitting of $\alpha$ (*i.e.*, finding a left inverse of it) is equivalent to splitting of $\beta$ (*i.e.*, finding a right inverse thereof). (We know in this case, $M \cong M' \oplus M''$.)

**Problem 4.** Calculate the Tor and Ext for a pair of any finitely generated abelian groups. Deduce that if $\operatorname{tor}(A)$ denotes the torsion subgroup (the subgroup of finite-order elements), then $\operatorname{Tor}(A, B) \cong \operatorname{tor}(A) \otimes \operatorname{tor}(B)$ and $\operatorname{Ext}(A, B) = \operatorname{tor}(A) \otimes B$.

**Problem 5.** Deduce the Mayer-Vietoris sequence from p. 149 of the text (actually, think of the terms there as homology with coefficients in a given $R$-module $M$) from the (unreduced) homology axioms for pairs of spaces, like those axioms we talked about in class.

**Problem 6.** Exercise 40 on p. 159 of Hatcher.

**Problem 7.** The $G \xrightarrow{n} G$ part of Exercise 2 on p. 204 of Hatcher. (Use the axioms/properties of Ext, rather than resolutions: represent $G$ as a quotient of a free abelian group).