Math 8306 **Fall 2021** **Homework 3**

Posted: 10/08; typo in Problem 3 corrected 10/14; due Friday, 10/15

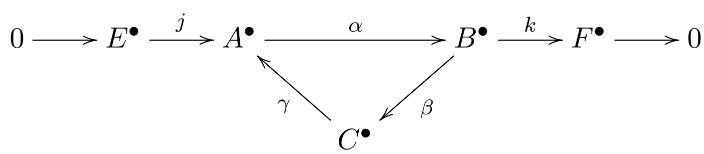
The problem set is due at the beginning of the class on Friday (on paper or by email). A *ring* R will always mean a commutative ring with unit, unless specified otherwise.

Reading:
Class notes.
Hatcher: Chapter 2 (pages 144, 160–165), Chapter 3 (pp. 190–202, 215), Section 3.A (pp. 263–267), Section 3.F (pp. 317–318).
Rotman's *An Introduction to Homological Algebra* (available online through the library): pp. 325–327, 340–343, 346, 349–350, 353–354, 370, 372]

Problem 1. Show that $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module, if I is a divisible abelian group. [*Hint*: Use the injectivity of I as an abelian group.]

Problem 2. Prove the splicing lemma: If $A \to B \xrightarrow{\alpha} C \to 0$ and $0 \to C \xrightarrow{\beta} D \to E$ are exact, then $A \to B \xrightarrow{\beta\circ\alpha} D \to E$ is exact.

Problem 3. Suppose



is a diagram of graded R-modules (e.g., $A^\bullet = \bigoplus_{n\in\mathbb{Z}} A^n$) with the top row exact and the triangle exact. An *exact triangle* here means (1) the degree of β is 1, meaning $\beta(B^n) \subseteq C^{n+1}$ for all n, whereas all the other maps are of degree 0, and (2) $\ker\beta = \operatorname{im}\alpha$, $\ker\gamma = \operatorname{im}\beta$, and $\ker\alpha = \operatorname{im}\gamma$. Prove that there is a short exact sequence

$$0 \to F^\bullet \xrightarrow{\beta k^{-1}} C^\bullet \xrightarrow{j^{-1}\gamma} E^\bullet \to 0$$

of graded R-modules (with $j^{-1}\gamma$ having degree 0 and βk^{-1} having degree 1).

Problem 4. Let $f : \mathbf{RP}^2 \to S^2$ be the map pinching the 1-skeleton of $\mathbf{RP}^2$ to a point. Compute the induced map on cohomology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2$ to see that the splitting of

$$0 \to \mathrm{Ext}_R(H_{n-1}(X, A; R), M) \to H^n(X, A; M) \to \mathrm{Hom}(H_n(X, A; R), M) \to 0$$

in the UCT for cohomology (for $R = \mathbb{Z}$, $M = \mathbb{Z}$ and $\mathbb{Z}/2$, $A = \emptyset$, and $X = \mathbb{RP}^2$ and S^2) to show that the splitting is not natural.

Problem 5 (A project worth 4 problems)**.** Study the 6-page excerpt from Greenberg and Harper's textbook (see the Homework page) and

(1) Formulate the acyclic models theorem;
(2) Formulate the Eilenberg-Zilber theorem;
(3) Deduce the Eilenberg-Zilber theorem from the acyclic models theorem.

Note that you will not need about 50% of the material of that excerpt.