Math 8306 **Fall 2021** **Homework 4**
Posted: 10/22; due Friday, 10/29

The problem set is due at the beginning of the class on Friday (on paper or by email).

Reading:
Class notes.
Hatcher: Chapter 3 (pp. 185–186, 191–196, 198–199, 206–213, 214–215, 239–241, 249), Section 3.A (pp. 264–265, 267), Section 3.B (pp. 268, 273–276, 277–280, Example 3B.4 on p. 272 for $X = Y = \mathbf{RP}^2$, the $G = \mathbb{Z}$, $m = n = 2$, homology part of Exercise 1 on p. 280).

A *ring* R will always mean a commutative ring with unit, unless specified otherwise. If there are no coefficients mentioned, for this homework, assume them to be R, unlike the common practice to assume them to be $\mathbb{Z}$. Same about $\otimes$ and Hom.

Problem 1. Compute the homology $H_\bullet(\mathbf{RP}^2 \times \mathbf{RP}^2; \mathbb{Z}/2)$ using our computation of the integral homology and the UCT for homology.

Problem 2. In class, we had a geometric interpretation of the generator of $H_3(\mathbf{RP}^2 \times \mathbf{RP}^2; \mathbb{Z}) = \mathbb{Z}/2$, which came from the Tor group in the Künneth formula. We did that using cellular homology. Give a geometric interpretation of of the generators of $H_1(\mathbf{RP}^2; \mathbb{Z}/2)$, $H_2(\mathbf{RP}^2; \mathbb{Z}/2)$, and $H^2(\mathbf{RP}^2; \mathbb{Z})$ which come through the universal coefficient theorems.

Problem 3. Show that for $\alpha, \beta \in H^\bullet(X)$ and $z \in H_\bullet(X)$,

$$\langle \alpha, \beta \cap z \rangle = \langle \alpha \cup \beta, z \rangle,$$

where

$$\langle -, - \rangle : S^\bullet(X) \otimes S_\bullet(X) \to R$$

denotes the Kronecker pairing, given on $S^p(X) \otimes S_p(X)$ by evaluation of singular cochains $S^p(X) = \mathrm{Hom}(S_p(X), R)$ on singular chains $S_p(X)$, and equal to zero, otherwise.

Problem 4. Directly from the definitions, compute the cellular cohomology groups of $S^1 \times S^1$ and the Klein bottle. (Time to appreciate how cumbersome simplicial homology is for computations!)

Problem 5. Problem 7 on p. 205 of Hatcher.

Problem 6. Problem 9 on p. 229 of Hatcher.

Problem 7. Problem 10 on p. 229 of Hatcher.

Problem 8. Problem 3 on p. 267 of Hatcher.