Math 8306 **Fall 2021** **Homework 5**
Posted: 11/5; Problem 4 simplified, due date changed: 11/10; due Monday, 11/15

The problem set is due at the beginning of the class on Monday (on paper or by email).

Reading:
Class notes.
Hatcher: Chapter 3 (pp. 203–204, 209–210, 240–241, 233–234, 236, 252–255, 249–250)

A *ring* R will always mean a commutative ring with unit, unless specified otherwise. If there are no coefficients mentioned, for this homework, assume them to be R, unlike the common practice to assume them to be $\mathbb{Z}$. Same about $\otimes$ and Hom.

Problem 1 (Counterexample for excisive pairs)**.** Let A be a point on a circle and B be the complement of A in the circle. Take $A \cup B$ with the standard topology of the circle. Show that $\{A, B\}$ is not an excisive pair.

Problem 2. Show that if X is covered by open, contractible sets U_i, $i = 1, \dots, n$, then $a_1 \cup \dots \cup a_n = 0$ for any collection of $a_i \in H^{n_i}(X)$ with $n_i > 0$. [*Hint*: Use the relative cup product.] (This would imply that the torus cannot be covered by two charts, given that the cup product of the two one-dimensional generators of cohomology is nontrivial, see below.)

Problem 3. Explain why the cap product

$$\cap : H^p(X, A) \otimes H_{p+q}(X, A) \to H_q(X)$$

is well-defined. (Not using the more general cap product

$$H^p(X, A) \otimes H_{p+q}(X, A \cup B) \to H_q(X, B),$$

which we did not back up, except for $A = \emptyset$.)

Problem 4. Deduce the following form of Alexander duality from Poincaré-Lefschetz duality: $H^p(K; \mathbb{Z}) \cong H_{n-p}(M, M \setminus K; \mathbb{Z})$ for a compact subspace K of a closed orientable n-manifold M, such that K has an open neighborhood $U \supset K$ in M so that U is homotopy equivalent to K and $M \setminus U$ is a manifold with boundary ∂U. [*Hint*: A more general case of K is done in Hatcher, but there is no need to do any limits or compact supports for this particular case.]

Problem 5. (1) For a compact, closed, oriented n-manifold M, show that the intersection pairing

$$H^p(M; \mathbb{Z}) \otimes H^{n-p}(M; \mathbb{Z}) \to \mathbb{Z}, \tag{1}$$

$a \cdot b = \langle a \cup b, [M] \rangle$, passes to a pairing

$$H^p(M; \mathbb{Z}) / \operatorname{Tor} \otimes H^{n-p}(M; \mathbb{Z}) / \operatorname{Tor} \to \mathbb{Z},$$

where Tor is the torsion subgroup.

(2) Show that the above pairing is perfect, *i.e.*, the adjoint

$$H^p(M;\mathbb{Z})/\operatorname{Tor} \to \operatorname{Hom}_{\mathbb{Z}}(H^{n-p}(M;\mathbb{Z})/\operatorname{Tor},\mathbb{Z})$$

is an isomorphism. [*Hint*: Use the UCT and PD.]

Problem 6. Show that $H^\bullet(T^n;\mathbb{Z})$ is an exterior algebra, using Poincaré duality and induction on n. Here $T^n = (S^1)^n$ is the n-torus.

Problem 7. Show that the form (1) for an even $n = 2k$ and $p = k$ is unimodular, that is to say, has determinant ± 1.