Math 8306 **Fall 2021** **Homework 6**
Posted: 11/19; Completed: 11/20; due Monday, 11/29

The problem set is due at the beginning of the class on Monday (on paper or by email).

Reading:
Class notes.
Hatcher: Chapter 3 (p. 250), Appendix A (pp. 523 and 529–531), Chapter 4 (pp. 375, 376–377, 379–380, 407–408), Section 4.H (pp. 460–461)

A *ring* R will always mean a commutative ring with unit, unless specified otherwise. If there are no coefficients mentioned, for this homework, assume them to be R, unlike the common practice to assume them to be $\mathbb{Z}$. Same about $\otimes$ and Hom.

Problem 1. Use the naturality of the cup product to show that the cohomology with the cup product is a homotopy invariant, *i.e.*, homotopy equivalent spaces have isomorphic cohomology R-algebras.

Problem 2. Let (X, x_0) and (Y, y_0) be two *pointed spaces* (*i.e.*, spaces with basepoints) and $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subseteq X \times Y$ be their *wedge product*. Show that cohomology algebra $H^\bullet(X \vee Y)$ is isomorphic to the product $H^\bullet(X) \times_R H^\bullet(Y) = \{(\alpha, \beta) \in H^\bullet(X) \times_R H^\bullet(Y) \mid \langle \alpha, x_0 \rangle = \langle \beta, y_0 \rangle\}$ of the augmented R-algebras.

Problem 3. Compute the cup-product on $H^\bullet(S^2 \vee S^4; \mathbb{Z})$ and show that $S^2 \vee S^4$ cannot be homotopy equivalent to $\mathbb{CP}^2$ despite having isomorphic integral homology groups.

Problem 4.
(1) Show that the trivial fibration $B \times F \to B$ is a fibration.
(2) Give an example to show that a homotopy lifting may not be unique.
(3) Show that the pullback fibration $f^*(E) \to A$ of a fibration $E \to B$ via $f : A \to B$ is a fibration.
(4) Show that the orthogonal projection of the right triangle onto one of its sides is a fibration. (This is an example of a fibration which is not a fiber bundle.)

Problem 5. Give an example of a surjective continuous map which is not a fibration and prove that it is not one directly by showing it fails the homotopy lifting property.

Problem 6. Prove that the pushout of a cofibration is a cofibration. (This is the dual statement of the problem on the pullback of a fibration above.)

Problem 7. Show directly, without appealing to fibrations, that the loop spaces of a path-connected space X at x and $y \in X$, $\Omega_x X$ and $\Omega_y X$, are homotopy equivalent.

Problem 8. Prove that if (X, A) and (Y, B) are cofibrations, then so is their product

$$(X, A) \times (Y, B) := (X \times Y, X \times B \cup A \times Y).$$