Math 8306 **Fall 2021** **Homework 7**
Posted: 12/8; Reading pages adjusted 12/13; due Wednesday, 12/15

The problem set is due at the beginning of the class on Wednesday (on paper or by email).

Reading:
Class notes.
Hatcher: Chapter 4 (pp. 409, 398–399, 395–396, 339–342, 376, 380, 349, 377–379, 381–382, 383–384, 341–342, 343–345)

Problem 1. Assume the unbased versions of the following based statements and prove the latter:

(1) If $F \to E \to B$ is a pointed fibration, then for any $Y \in K_*$ (the category of nondegenerately pointed spaces), the sequence
$$[Y, F]_0 \to [Y, E]_0 \to [Y, B]_0,$$
where $[-, -]_0$ stands for the set of based homotopy classes of based maps, is exact. Beware that in the unbased version, B was assumed to be path-connected.

(2) If $A \to X \to X/A$ is a pointed cofibration, then for any $Y \in K_*$, the sequence
$$[X/A, Y]_0 \to [X, Y]_0 \to [A, Y]_0$$
is exact. Beware that in the unbased version, Y was assumed to be path-connected.

Problem 2. Show that if $X \in K_*$, then the quotient map $\Sigma X \to SX$ from the unreduced suspension to the reduced one is a homotopy equivalence.

Problem 3. Prove that $\pi_n(X \times Y) = \pi_n(X) \oplus \pi_n(Y)$.

Problem 4. Prove the associativity of the smash product. Deduce $SS^n = S^{n+1}$ for $n \geq 0$, where $SX = S^1 \wedge X$ is the reduced suspension.

Problem 5. Show that $\pi_1(U) = \mathbb{Z}$ and $\pi_2(U) = 0$. Here $U = U(\infty) = \bigcup_n U(n)$ is the infinite unitary group.

Problem 6. Use the long exact homotopy sequence for a fibration to prove that $\mathbf{CP}^\infty = \bigcup_n \mathbf{CP}^n$ is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2)$, *i.e.*, a CW complex with the only nonzero homotopy group being $\pi_2 \cong \mathbb{Z}$.

Problem 7. Hatcher in proving the long exact sequence of a fibration $F \hookrightarrow E \to B$, Theorem 4.41, basically defines the connecting homomorphism $\pi_n(B) \to \pi_{n-1}(F)$ via the connecting homomorphism of the long exact sequence of the pair (E, F). We will have defined them separately, the map $\pi_n(B) \to \pi_{n-1}(F)$ coming from the Puppe sequence
$$\Omega^n B \to \Omega^{n-1} F \to \Omega^{n-1} E \to \Omega^{n-1} B.$$
Prove that these two constructions give one and the same map.

Problem 8. Show that a topological group is always a simple (a.k.a. abelian, e.g., in Hatcher) space, that is to say, π_1 acts trivially on each π_n.