MATH 8306: ALGEBRAIC TOPOLOGY
PROBLEM SET 1, DUE FRIDAY, SEPTEMBER 27

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Section 2.3 (from Hatcher's textbook): 1, 2, 3, 4 (Do #4 for nonreduced homology and the wedge axiom replaced by the additivity axiom)
Appendix A, p. 529: 2, 3, 4

Problem 1. Let X_1 and X_2 be two filled doughnuts, $f : \partial X_1 \to \partial X_2$ a homeomorphism, $M^3_f = X_1 \cup_f X_2$, Find such homeomorphisms f, that M_f is homeomorphic to S^3, $S^2 \times S^1$, and $\mathbb{RP}^3$.

Problem 2. Let X be a topological space and x_0, x_1 given points in it. Let Y be the space of paths starting at x_0 and *passing* through x_1. Show that Y is contractible. [Hint: This problem is not a misprint, although may seem wrong on the first sight.]

Problem 3. Use the Mayer-Vietoris sequence to give another derivation of the homology groups of spheres.

Problem 4. Use the Mayer-Vietoris sequence to compute the homology of the space which is the union of three n-disks along their common boundaries.

Problem 5. Prove the following statements, using the homology axioms. If A and B are subspaces of a topological space X and $X = A \cup B$ such that the excision homomorphisms

$$H_\bullet(A, A \cap B) \to H_\bullet(X, B)$$
$$H_\bullet(B, A \cap B) \to H_\bullet(X, A)$$

are isomorphisms, then the Mayer-Vietoris sequence

$$\cdots \to H_p(A \cap B) \xrightarrow{\phi} H_p(A) \oplus H_p(B) \xrightarrow{\psi} H_p(X) \xrightarrow{\partial_p} H_{p-1}(A \cap B) \to \dots,$$

where ϕ is the sum of the homomorphisms induced by the inclusions, ψ is the difference of those induced by the inclusions, and ∂ is a certain connecting homomorphism, is exact.

Problem 6. Calculate the homology of $S^p \vee S^q$ using the Mayer-Vietoris sequence.

Problem 7. Represent a compact orientable surface of genus g (i.e., the sphere with g handles or a torus with g holes) as a CW complex.

Problem 8. Is the subset $\bigcup_{n=1}^{\infty} \{(x, y) \mid x^2 - x/n + y^2 = 0\}$ of $\mathbb{R}^2$ a CW complex?

Problem 9. For a cell complex X, show that

$$H_p(X^n, X^{n-1}) = 0, \qquad p \neq n,$$

where X^q denotes the q-skeleton of X.

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Problem 10. Use the integral homology of the real projective plane $\mathbb{RP}^2$,

$$H_n(\mathbb{RP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}_2 & \text{for } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

to compute the homology $H_\bullet(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$ of the product space $\mathbb{RP}^2 \times \mathbb{RP}^2$.

Problem 11. Compute the homology of $\mathbb{RP}^2$ over $\mathbb{Z}_2$ using its integral homology groups.