MATH 8306: ALGEBRAIC TOPOLOGY
PROBLEM SET 2, DUE MONDAY, NOVEMBER 4

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Problem 1. Give an example of two CW complexes which have the same homology but are not homotopy equivalent.

Problem 2. Show that every self-map of $\mathbb{RP}^{2n}$ has a fixed point.

Problem 3. Let A be a subcomplex of a CW complex X, Y a CW complex, $f : A \to Y$ a cellular map, and $Y \cup_f X$ the pushout (i.e., the attaching space). The Euler characteristic of a finite CW complex X may be defined as $\chi(X) = \sum_n (-1)^n \gamma_n(X)$. Formulate and prove a formula relating the Euler characteristics of A, X, Y, and $X \cup_f Y$, when X and Y are finite.

Problem 4. Let $p : X \to Y$ be a covering map with finite fibers of cardinality n. Using singular chains, construct a homomorphism $t : H_\bullet(Y; G) \to H_\bullet(X; G)$ such that the composite $p_* \circ t : H_\bullet(Y; G) \to H_\bullet(Y; G)$ is a multiplication by n. [t is called a *transfer homomorphism*.]

Problem 5. Using cell decompositions of S^n and D^n, compute their cellular homology groups.

Problem 6. Calculate the integral homology groups of the sphere M_g with g handles (*i.e.*, a compact orientable surface of genus g) using a cell decomposition of it.

Problem 7. Prove the "coassociativity" property $(\mathrm{id} \otimes \mathrm{AW}) \circ \mathrm{AW} = (\mathrm{AW} \otimes \mathrm{id}) \circ \mathrm{AW}$ of the Alexander-Whitney map $\mathrm{AW} : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$.

Problem 8. Give a simple proof that uses homology theory for the following fact: $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for $m \neq n$.

Problem 9. Regarding a singular cochain $\phi \in C^1(X; G)$ as a function from paths in X to G, show that if ϕ is a cocycle, that is, $\delta\phi = 0$, then

1. $\phi(f \cdot g) = \phi(f) + \phi(g)$,
2. ϕ takes the value 0 on constant paths,
3. $\phi(f) = \phi(g)$ if $f \sim g$ via a homotopy fixing the endpoints,
4. ϕ is a coboundary (that is, $\phi = \delta\psi$ for some $\psi \in C^0(X; G)$) iff $\phi(f)$ depends only on the endpoints of f, for all f.

[In particular, 1 and 4 give a homomorphism $H^1(X; G) \to \mathrm{Hom}(\pi_1(X), G)$, which is a version of Hurewicz isomorphism if X is path connected.]

Problem 10. Assuming as known the cup product structure on the torus $T^2 = S^1 \times S^1$ ($\langle e^0 \rangle^* \cup a = a$ for a generator $\langle e^0 \rangle^*$ of $H^0(T; \mathbb{Z}) = \mathbb{Z}$ and any $a \in H^\bullet(T; \mathbb{Z})$, $\langle e^1_1 \rangle^* \cup \langle e^1_2 \rangle^* = \langle e^2 \rangle^*$ and $\langle e^1_i \rangle^* \cup \langle e^1_i \rangle^* = 0$ for $i = 1, 2$ and generators $\langle e^1_1 \rangle^*$ and

Date: October 19, 2002.

$\langle e_2^1\rangle^*$ of $H^1(T;\mathbb{Z})$ coming from two natural projections $T \to S^1$ via pullback, and $\langle e^2\rangle^* \cup \langle e_i^1\rangle^* = \langle e^2\rangle^* \cup \langle e^2\rangle^* = 0$), compute the cup product in $H^\bullet(M_g;\mathbb{Z})$ for the compact orientable surface M_g of genus g, by using a quotient map from M_g to the bouquet of g tori.

Problem 11. Show that any map $S^4 \to S^2 \times S^2$ must induce the zero homomorphism on $H_4(-)$. [Hint: Use the cup product]

Problem 12. Prove that there is no homotopy equivalence $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ that reverses orientation (induces multiplication by -1 on $H_{4n}(\mathbb{CP}^{2n};\mathbb{Z})$). [Hint: note from cellular cohomology that the generator $\langle e^{4n}\rangle^* \in H^{4n}$ is a cup square $\langle e^{2n}\rangle^* \cup \langle e^{2n}\rangle^*$, which you may assume is true. To pass from cohomology to homology, use the naturality of the Universal Coefficients Theorem.]