MATH 8306: ALGEBRAIC TOPOLOGY
PROBLEM SET 3, DUE FRIDAY, DECEMBER 13

SASHA VORONOV

Problem 1. Let $X = \Sigma Y = Y \wedge S^1$ be a reduced suspension. Show that the cup product $\tilde{H}^p(X) \otimes \tilde{H}^q(X) \to \tilde{H}^{p+q}(X)$ is the zero homomorphism, where the coefficients are assumed to be in a commutative ring R and the tensor product is over R. [Hint: For a pointed space X and two open subspaces A and B with a basepoint $* \in A \cap B$, construct a commutative diagram

$$\begin{array}{ccc} H^p(X,A) \otimes H^q(X,B) & \longrightarrow & H^{p+q}(X, A \cup B) \\ \downarrow & & \downarrow \\ \tilde{H}^p(X) \otimes \tilde{H}^q(X) & \longrightarrow & \tilde{H}^{p+q}(X) \end{array}$$

and use it in the case $X = A \cup B$, where A and B are contractible.]

Problem 2. Let M be a compact orientable n-manifold. Suppose that M is homotopy equivalent to ΣY for some connected pointed space Y. Deduce that M has the same integral homology as S^n.

Problem 3. Prove that for $n \le \infty$, $H^\bullet(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[t]/(t^{n+1})$ as a ring, with degree $|t| = 1$. [Hint: This is a repetition of what we did in class for $\mathbb{CP}^n$, plus a new statement for $n = \infty$, where Poincaré duality does not work directly.]

Problem 4. Compute the de Rham cohomology of S^1 from definitions. [Hint: It might be easier to think of S^1 as a quotient manifold $\mathbb{R}/\mathbb{Z}$, rather than as something glued out of two open intervals, and use the "angular" coordinate x, the one coming from $\mathbb{R}$.]

Problem 5. Prove that any CW complex X has *numerable category*, *i.e.*, admits a numerable cover $\{U_j\}$ such that each inclusion map $U_j \hookrightarrow X$ is null-homotopic. [You may assume that CW complexes are paracompact.]

Problem 6. Let

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ F' & \longrightarrow & X' & \longrightarrow & Y' \end{array}$$

be a homotopy commutative diagram in which the rows are homotopy fibrations and X', Y, and Y' have the homotopy type of connected CW complexes. Show that if two of the vertical maps are homotopy equivalences, then so is the third.

Problem 7. Let $p : X \to B$ and $q : Y \to B$ be fibrations and $f : X \to Y$ be a fibration map, *i.e.*, satisfy $qf = p$. Prove that if f is a homotopy equivalence, then f is a fiber homotopy equivalence. [Hint: Dualize the proof of Proposition 0.19 from Hatcher.]

Date: December 4, 2002.

Problem 8. Prove that the inclusion of a subcomplex A into a CW complex X is a cofibration.

Problem 9. Show that a map $A \to X$ satisfying the Homotopy Extension Property is automatically an inclusion (in particular, a cofibration) with closed image. Assume that X is Hausdorff.

Problem 10. Let $A \hookrightarrow X$ be a cofibration, where A is contractible. Prove that the quotient map $X \to X/A$ is a homotopy equivalence.