MATH 8307: ALGEBRAIC TOPOLOGY
PROBLEM SET 4, DUE FEBRUARY 24, 2003

SASHA VORONOV

Section 4.1 (from Hatcher's textbook): 15 (You may use smooth approximation in (a), if you wish so.), 18
Section 4.2 (from Hatcher's textbook): 15 (In fact, prove a more general statement that a closed connected simply-connected n-dimensional manifold whose homology is the same as that of S^n is homotopy equivalent to S^n.) [Hint: use the Hurewicz theorem.]

Problem 1. Show that, if $n \geq 2$, then $\pi_n(X \vee Y)$ is isomorphic to

$$\pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

Problem 2. Compute $\pi_n(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ for $n \geq 2$. Deduce that the quotient map

$$(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \to (\mathbb{RP}^n/\mathbb{RP}^{n-1}, *)$$

does not induce an isomorphism of homotopy groups.

Problem 3. Compute the homotopy groups of $\mathbb{CP}^n$ in terms of the homotopy groups of spheres.

Problem 4. Compute all of the homotopy groups of $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$.

Problem 5. Show that $\pi_7(S^4)$ contains an element of infinite order. Do not use any theorems we have not proven. [Hint: Use the Hopf bundle and the fact that $\pi_3(S^7) = 0$.]

Problem 6. Assume given maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$. Suppose that Y is a CW complex. Show that X has the homotopy type of a CW complex, *i.e.*, is homotopy equivalent to a CW complex.

Problem 7. Let $n \geq 1$ and π be an abelian group. Construct a connected CW complex X such that $\widetilde{H}_n(X; \mathbb{Z}) = \pi$ and $\widetilde{H}_q(X; \mathbb{Z}) = 0$ for $q \neq n$. Such space X is denoted $M(\pi, n)$ and called a *Moore space*. [Hint: construct $M(\pi, n)$ as the cofiber of a map between wedges of spheres.]

Problem 8. Let $n \geq 1$ and π be an abelian group. Construct a connected CW complex X such that $\pi_n(X) = \pi$ and $\pi_q(X) = 0$ for $q \neq n$. Such space X is denoted $K(\pi, n)$ and called an *Eilenberg-Mac Lane space*. [Hint: start with $M(\pi, n)$, use the Hurewicz theorem, and kill the higher homotopy groups.]

Date: February 8, 2003.