MATH 8307: ALGEBRAIC TOPOLOGY
PROBLEM SET 5, DUE APRIL 7, 2003

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Problem 1. Determine the integral homology of the total space of a fibration over S^2 with fiber S^2.

Problem 2. Consider the cohomology spectral sequence of a three-sphere bundle $U(2)/U(1) = S^3 \to BU(1) \to BU(2)$ with coefficients in $\mathbb{R}$. Show that $E_2^{p,3} = E_4^{p,3}$ and $E_2^{p+4,0} = E_4^{p+4,0}$. Using $H^\bullet(BU(1); \mathbb{R}) \cong \mathbb{R}[c_1]$, $c_1 \in H^2(BU(1); \mathbb{R})$, show that $d_4 : E_2^{p,3} \to E_2^{p+4,0}$ is injective for $p \geq 0$.

Problem 3. For the Leray-Serre cohomology spectral sequence of a fibration $S^n \to E \xrightarrow{\pi} B$ over a simply connected base B, show that there exists an element $\Omega \in H^{n+1}(B; \mathbb{R})$, such that the map $\Psi : H^p(B; \mathbb{R}) \to H^{p+n+1}(B; \mathbb{R})$ defined by $\Psi(u) = u \cup \Omega$, $u \in H^p(B; \mathbb{R})$, induces the following exact sequence

$$\cdots \to H^{p+n}(E; \mathbb{R}) \to H^p(B; \mathbb{R}) \xrightarrow{\Psi} H^{p+n+1}(B; \mathbb{R}) \xrightarrow{\pi^*} H^{p+n+1}(E; \mathbb{R}) \to \dots$$

This is called the *cohomology Gysin sequence* of a sphere fibration.

Problem 4. Show that for $n \geq 1$

$$H_\bullet(\Omega^2 S^{2n+1}; \mathbb{Z}_p) \cong \bigotimes_{k=0}^{\infty} \mathbb{Z}_p[u_{2np^k-1}] \otimes \bigotimes_{k=1}^{\infty} \mathbb{Z}_p[v_{2np^k-2}]$$

as a $\mathbb{Z}_p$-vector space, where p is a prime and $\mathbb{Z}_p[x_m]$ denotes the graded symmetric algebra on an element of degree m. (So that when m is even, we have a polynomial algebra, and when m is odd, we have an exterior algebra). [**Hint**: Use the *cohomological* Leray-Serre spectral sequence for the fibration $\Omega^2 S^{2n+1} \to P\Omega S^{2n+1} \to \Omega S^{2n+1}$ and the computation of the cohomology ring $H^\bullet(\Omega S^{2n+1}; \mathbb{Z})$ as the divided polynomial ring $\mathbb{Z} \oplus \mathbb{Z}x \oplus \dots \oplus \mathbb{Z}x^i/i! \oplus \dots$ in an element $x \in H^{2n}(S^{2n+1}; \mathbb{Z})$, see p. 224 of Hatcher, where the cohomology ring of the James construction $J(S^{2n}) \sim \Omega\Sigma S^{2n} = \Omega S^{2n+1}$ is computed. To compute the first nonzero differential d_{2n}, note that $d_{2n} : E_2^{0,2n-1} \to E_2^{2n,0}$ must be an isomorphism, which implies there exists an element $u \in H^{2n-1}(\Omega^2 S^{2n+1}; \mathbb{Z}_p)$ such that $d_{2n}(1 \otimes u) = x \otimes 1$. Use the derivation property to compute $d_{2n}((x^i/i!) \otimes u)$. Show that $\{1, u\}$ is a basis of $H^\bullet(\Omega^2 S^{2n+1}; \mathbb{Z}_p)$ in dimensions $< 2pn - 2$. Proceed in this fashion. See that thiscontinues to the right for p steps, then breaks, creating a new pattern, which continues to the right for p steps, creating another pattern, and so on. After you have computed the cohomology, use the universal coefficients formula to compute the homology.]

Problem 5. Consider an Eilenberg-Mac Lane space $K(\mathbb{Z}, 3)$. Let $j : S^3 \to K(\mathbb{Z}, 3)$ be a generator of $\pi_3(K(\mathbb{Z}, 3))$ and $S\langle 3\rangle$ denote the homotopy fiber of j, i.e., the

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fiber of the mapping path fibration of j. Consider the fiber sequence

$$\cdots \to \Omega K(\mathbb{Z}, 3) \to S^3\langle 3\rangle \to S^3 \to K(\mathbb{Z}, 3).$$

How do the homotopy groups of $S^3\langle 3\rangle$ compare to those of S^3? Note that $\Omega K(\mathbb{Z}, 3) = K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$ up to homotopy equivalence. Use the Leray-Serre spectral sequence for the homotopy fibration $\mathbb{CP}^\infty \to S^3\langle 3\rangle \to S^3$ to calculate $H_\bullet(S^3\langle 3\rangle)$ and use the result to show that $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. [**Hint**: Use the cohomological Leray-Serre spectral sequence and the derivation property to compute d_3 explicitly, using the cohomology ring of $\mathbb{CP}^\infty$. Then go back to homology by UCT.]

Problem 6. Read the definition of a Thom class and the Thom space of a disk bundle in Hatcher, p. 441 of a hardcopy. Prove the Thom isomorphism theorem, Corollary 4D.9, using the Leray-Serre spectral sequence for an S^n-bundle you associate with the D^n-bundle (neither the Leray-Hirsch theorem, which is Theorem 4D.8, nor the Mayer-Vietoris sequence).